



Asymptotic Shapley Value of Games with Large and Small Players

Anna Sabak

Department of Mathematical Economics, Warsaw School of Economics, Poland
(e-mail: asaba@sgh.waw.pl)

Abstract This paper deals with cooperative games with large and small players. Large players are distinguishable individuals (there is a finite number of them in a game) and small players are considered jointly and represented by a set of power of continuum, which can be divided into a finite number of subsets (types of small players). The generalised version of the glove-market game, involving large and small players (called here the minimum game), and its counterpart, the maximum game, are introduced. For these games, the formulae for asymptotic Shapley value are calculated and some properties are identified, in particular, an asymmetry of large and small players. The cases when the value is independent of the number of small players, and when the value belongs to the core are described.

JEL Classifications C71

Keywords games with large and small players, glove-market game, (asymptotic) Shapley value

1. Introduction

We investigate cooperative games with two kinds of players: distinguishable individuals known as ‘large’ players (there is a finite number of them in a game) and ‘small’ players, considered jointly and represented by a set of power of continuum. Games with continuum of small players are described by Aumann and Shapley (1974), who introduced the notion of the asymptotic (Shapley) value for non-atomic games. The asymptotic Shapley value is obtained for a class of games by calculating the value for an n -person game and taking the limit of the value when the number of players tends to infinity – this asymptotic approach was also used by Kannai (1966). Other approaches existing in literature include: the potential approach applied by

Hart and Monderer (1997) to weighted values of non-atomic games and the random order approach developed by Santos and Zarzuelo (1998). The existence and properties of values for games with both large and small players (also called *mixed games*) were examined by Hart (1973). In this paper we apply the asymptotic approach to mixed games.

We discuss games in which small players can differ from each other. In this case their set is divided into a finite number of subsets, each representing one type of small players. Such games will be called *multi-type mixed games*. This approach to cooperative games with continuum of players enables a closer examination of such games than it is possible in the general setup of mixed games. The introduction of large and small players takes into account the fact that differences in the significance of the players in the same game are possible. The division of the set of small players into a finite number of types is due to the fact that small players are not necessarily identical. It also makes investigating properties of games simpler.

A generalised version of the glove-market game (also called the *minimum game*) is investigated. Large and small players possess different goods and they individually decide to use their commodity (whole amount or just a part of it). The payoff of the coalition is equal to the value of the smallest input of a participant. We interpret this situation as a production process, where the large players and the types of small players are the suppliers of raw materials used in fixed proportions.

A counterpart of the minimum game, a *maximum game* is also examined. Here the payoff is the maximum of inputs of all agents (large players or types of small players). As an example of such game we consider actions aimed at improving social health, taken by the authorities (large players) and individuals (small players). The government decides whether to launch a large-scaled programme, while individuals make decisions about their (more healthy or less) lifestyles. The payoff is the amount of money (either public or private) invested to good effect.

For both the minimum and the maximum games, the formulae for the asymptotic Shapley values are calculated in a restricted case of one large player and one type of small players. Relations between the values and the core are investigated, together with some other properties of the value.

The paper is organised as follows. Section 2 contains a definition of a multi-type mixed game with large and small players. We briefly recall the basic concepts of theory of cooperative mixed games, such as coalitions, imputations, core, etc., introduced by Sabak and Wiczorek (2003) analogously to the respective terms for n -person games, but regarding the specific form of the set of players. In this section we also discuss the asymptotic Shapley value for the multi-type games. The generalised glove-market game,

formulae for its asymptotic Shapley value and the relation of the value and the core are described in section 3. Section 4 is dedicated to the maximum game and its properties. We conclude in section 5.

2. Basic concepts

We deal with cooperative games of n large players and n' types of homogeneous small players. We assume that $n+n'$ is positive, if $n=0$, we have only small players, if $n'=0$, we have only large players. A *coalition* is defined as an element of the set $\mathbf{C} = \{0,1\}^n \times [0,1]^{n'}$. A coalition of the form $C = (\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\alpha^1, \dots, \alpha^n; \beta^1, \dots, \beta^{n'})$ is interpreted as one consisting of large players who belong to $\{i=1, \dots, n \mid \alpha^i = 1\}$ and fractions of small players of all types, where the fraction of those of type k is equal to β^k .

In this setup, the *zero coalition* is represented by

$$\mathbf{0}_{n,n'} = (\underbrace{0, \dots, 0}_n; \underbrace{0, \dots, 0}_{n'})$$

while the *grand coalition* is represented by

$$\mathbf{1}_{n,n'} = (\underbrace{1, \dots, 1}_n; \underbrace{1, \dots, 1}_{n'})$$

If no confusion can possibly arise, we shall abbreviate $\mathbf{0}_{n,n'}$ into $\mathbf{0}$ and $\mathbf{1}_{n,n'}$ into $\mathbf{1}$.

2.1 Multi-type mixed game

Given the number of large players n and the number of types of small players n' , we define a *cooperative game* (also called a *characteristic function*) as a function $v: \mathbf{C} \rightarrow R$, such that $v(\mathbf{0}) = 0$; so a game assigns a real number to each coalition. A cooperative game with the set of players and characteristic function defined in this way will be called a *multi-type (mixed) game*.

A *pre-imputation* in a game v is a sequence $\mathbf{x} = (x^1, \dots, x^n; \xi^1, \dots, \xi^{n'})$ where all x^i are real numbers, while all ξ^i are non-decreasing left-hand side continuous real functions on $[0,1]$, such that

$$\sum_{i=1}^n x^i + \sum_{i=1}^{n'} \int_0^1 \xi^i(t) dt = v(\mathbf{1})$$

Example Let us consider a 2-person 3-type game ν , such that

$$v(\alpha^1, \alpha^2; \beta^1, \beta^2, \beta^3) = \alpha^1 + \alpha^2 + \frac{\beta^1 + \beta^2 + \beta^3}{3}$$

We have $v(\mathbf{1}) = 3$. Let us take a sequence $\mathbf{x} = (x^1, x^2; \xi^1, \xi^2, \xi^3)$, where $t \in [0, 1]$ and

$$x^1 = \frac{2}{5}, \quad x^2 = \frac{3}{5}, \quad \xi^1 \equiv 1, \quad \xi^2(t) = 2t - 1, \quad \xi^3(t) = \begin{cases} -3 & \text{if } 0 \leq t \leq \frac{1}{3} \\ 3 & \text{if } \frac{1}{3} < t \leq 1 \end{cases}$$

It can be easily verified that \mathbf{x} is a pre-imputation in the game ν .

Small players within the same type are homogeneous – when a coalition is created, the only factor affecting its payoff is the size of the fraction of small players joining the coalition. However, when it comes to sharing the payoff among the players of the type, we allow unequal (non-uniform) distribution, because not all players necessarily have the same access to the payoff. A pre-imputation is understood as a sort of distribution of the total payoff of the grand coalition among all players. So in terms of our model, we do not restrict ξ^i s to be constant functions, but we use the non-decreasing ones instead. In other words, we assume that no *a priori* differentiation of the small players within the type exists, but we permit differences in the players' wealth, resulting from the non-uniform distribution of the payoff of the coalition.

In the light of the remarks above, x^i is the share of i -th large player ($i = 1, \dots, n$) and $\xi^i(t)$, where $t \in [0, 1]$ and $i = 1, \dots, n'$, denotes the 'maximal (infinitesimal) payoff in the least wealthy fraction of size t of players of type i' '.

The notions of the rationality of pre-imputations (individual, considered separately with respect to large and small players, and coalitional) and relation of dominance, as well as the definitions of imputations and the core in the setup of multi-type games, are derived from the analogous concepts for n -person games and introduced in the paper by Sabak and Wieczorek (2003). For the purpose of this paper, we just recall the theorem describing the core:

Theorem 1 (Sabak and Wieczorek 2003). For any game ν , the imputation \mathbf{x} belongs to the core $C(\nu)$ if and only if for every coalition (α, β)

$$\sum_{\substack{i=1,\dots,n \\ \alpha^i=1}} x^i + \sum_{i=1,\dots,n'} \int_0^{\beta^i} \xi^i(t) dt \geq v(\alpha, \beta) \quad (1)$$

This theorem states that the core only consists of the pre-imputations which are individually rational w.r.t. both large and small players (we call them *imputations* in that case), and which are coalitionally rational (condition (1) holds for every coalition (α, β)).

2.2 Asymptotic Shapley value

For multi-type games, we are interested in the Shapley value, as it is the measure reflecting the relative importance of players. There is a direct formula, introduced by Shapley (1953), for the value of each player in the case of cooperative n -person games. However, we are not able to use it while dealing with games with large and small players, because there is an infinite number of players in such games. Instead, we propose the calculation of an asymptotic Shapley value (compare Aumann and Shapley 1974). Firstly, we consider the $(n + n')$ -person game with n players called 'large' and n' players called 'small', although there is no actual difference between them – all of them are equally important participants in a standard multi-person game. Each 'type of small players' is represented here by just one player. Secondly, we take a sequence of games, where every subsequent game has the same number of 'large' players and a greater number of 'small' players in each type than the previous game. We assume that the joined importance of the 'type of small players' (measured, for example, by the type's total weight in the voting games) remains the same. For each game in such a sequence, we find its classical Shapley value, obtaining a sequence of (ever longer) vectors. We take the limit of the sequence of values for a large player as his asymptotic Shapley value. To calculate the value of a type we take the sum of the values of the small players of this type in each game, and then find the limit. This procedure was used for simple games by Dubey and Shapley (1979) and it can be applied for other kinds of games as well. For the game with n large players and n' types of small players, we obtain the $(n + n')$ -vector φ , which we call the *asymptotic Shapley value for multi-type games*.

The value defined this way, however, lacks one important property, when compared with the Shapley value for n -person games, namely, as a vector it cannot be a pre-imputation, which in our setup is a sequence of real numbers and real functions (see section 2.1). Therefore, in order to investigate the relation between the value φ and the core of the game, let us introduce a pre-

imputation φ_c *connected* to the value. We consider a sequence of n numbers equal to the asymptotic values for the large players: $x^i = \varphi^i$, $i = 1, \dots, n$ and n' non-decreasing functions on $[0, 1]$, satisfying the condition:

$$\int_0^1 \xi^i(t) dt = \varphi^i, \quad i = 1, \dots, n'$$

Such a definition implies that an infinite number of possible functions can be considered.

3. Generalised glove-market game

In a glove-market game there are n players (traders) and m goods. The player i possesses an amount $b_j^i \geq 0$ of a commodity j , where $i = 1, \dots, n$, $j = 1, \dots, m$. On the market there is a demand for equal amounts of goods, and we assume that each commodity is being sold at a unit price. A trader does not have to possess each commodity, so in order to sell his goods, he engages in coalitions with other traders. The payoff for a coalition $S \subseteq \{1, \dots, n\}$ is

$$v(S) = \min \left\{ \sum_{i \in S} b_j^i \mid j = 1, \dots, m \right\}$$

that is, a value of the commodity held together by the coalition in the smallest amount.

The generalised non-atomic version of the n -person market-glove game was proposed by Sabak and Wieczorek (2003). Standard players are replaced there by n' types of small players, and there are n' kinds of goods. It is assumed that each type of small players possesses only one commodity. The value of the characteristic function is equal to the minimum of the fractions of the types participating in the coalition. This corresponds to the situation of n' groups of cooperating producers. In order to manufacture a certain amount of goods, it is necessary to use parts and materials delivered by those producers in appropriate proportions as input (a possible excess is wasted).

Here, let us investigate the mixed generalised version of the glove-market game. We introduce large and small players who are the owners of different kinds of 'gloves' or, rather, of different commodities. We assume that each large player and each type of small players is the sole owner of its commodity. The game is interpreted as an interaction between suppliers of inputs in the fixed proportions production (of the Leontief type) in which the amount of the produced good is equal to the size of the smallest input. The payoff in the

minimum game is equal to the amount of the produced commodity. Large players are producers who possess machines, devices, trucks, etc. When such a producer decides to begin the activity, he has to engage all his possessions – you cannot produce using only a part of your production line. Small players are the suppliers of different types of raw materials and small players of the same type possess the same commodity. They work independently and possibly not all of them decide to participate in the enterprise, what can result in a delivery of only fractions of the attainable amounts of materials.

There are n large players and n' types of small players and each of them is assigned a positive real number, L^1, \dots, L^n and $R^1, \dots, R^{n'}$, respectively. At the level of interpretation, these numbers describe amounts (values) of different commodities held by agents (the amount owned by a certain type of small players being equally divided among its members), and correspond to the left and right gloves possessed by the players in a standard game. Let us highlight that the commodities held by the large players (or the types of small players) are different and cannot be added, that is 'left one' glove of the first large player is not the same as 'left two' glove of the second large player, and they cannot be compared in terms other than their value.

A coalition is of the form: $C = (\alpha, \beta) = (\alpha^1, \dots, \alpha^n; \beta^1, \dots, \beta^{n'})$ where $\alpha^i \in \{0, 1\}$ for $i = 1, \dots, n$ and $\beta^i \in [0, 1]$ for $i = 1, \dots, n'$. The value of the characteristic function for the coalition (α, β) is:

$$v(\alpha, \beta) = \min\{\alpha^1 L^1, \dots, \alpha^n L^n; \beta^1 R^1, \dots, \beta^{n'} R^{n'}\} \quad (2)$$

To make calculations of an asymptotic Shapley value easier, let us consider the case with one large player and one type of small players. We have the sequence of $(k+1)$ -person games, in which there is always one large player and the number of small players k tends to infinity. Amounts possessed by agents are constant and equal L for the large player and R for the type of small players. R is equally divided among the growing number of players who become smaller and smaller – in the sense of the amount they individually have at their disposal.

A coalition in the $(k+1)$ -person game is of the form:

$$C = (a; b^1, \dots, b^k)$$

where $a, b^1, \dots, b^k \in \{0, 1\}$. The value of the characteristic function is:

$$v(a, \mathbf{b}) = \min\left\{aL, \sum_{i=1}^k \frac{R}{k} b^i\right\} \quad (3)$$

To measure the relative power, represented by the asymptotic Shapley value, of the large player and the type of small players, we have the following:

Theorem 2 Let v be a generalised glove game with one large player and one type of small players, who possess amounts of L and R , respectively. The asymptotic Shapley value φ of the game v is as follows:

1. $\varphi = \left(\frac{R}{2}, \frac{R}{2} \right)$ when $L \geq R$
2. $\varphi = \left(\frac{L}{2R}(2R-L), \frac{L^2}{2R} \right)$ when $L < R$

Proof Let us recall the formula for the Shapley value of n -person games:

$$\varphi^i = \frac{1}{n!} \sum_{\substack{S \ni i \\ S \subset N}} s!(n-s-1)! [v(S) - v(S-i)] \quad (4)$$

where $n = |N|$ is the number of all players, $s = |S|$ is the number of players in a coalition S and i denotes the specific player, for whom the value is calculated. Applying formula (4) to our case leads to the expression dependent on the number of small players k . Then we let k tend to infinity and calculate the limit. Let us consider separately cases $L \geq R$ and $L < R$, as these turn out to be non-symmetric.

Case 1 $L \geq R$ We recalculate formula (4) for the large player:

$$\varphi_k^L = \frac{k!}{(k+1)!} \frac{R}{k} \left[\sum_{i=1}^k i \right] = \frac{R}{2} = \varphi^L$$

We can see that φ_k^L does not depend on the number of small players k . Analogously, we obtain the Shapley value for one small player j , $j = 1, \dots, k$:

$$\varphi_{k,j}^R = \frac{1}{(k+1)!} \frac{R}{k} \left[\sum_{r=0}^{k-1} (r+1)!(k+1-r-1-1)! \binom{k+1}{r} \right] = \frac{R}{2k}$$

We take the sum of values of small players as the value of the type. Using the fact that small players within one type are identical, we get:

$$\varphi_k^R = k\varphi_{k,j}^R = \frac{R}{2} = \varphi^R$$

In this way we obtain the vector of Shapley value for the considered case:

$$\varphi = (\varphi^L, \varphi^R) = \left(\frac{R}{2}, \frac{R}{2} \right) \quad (5)$$

Case 2 $L < R$ Firstly, we define

$$m_k = \left\lfloor \frac{L}{R/k} \right\rfloor = \left\lfloor \frac{kL}{R} \right\rfloor$$

This number tells us how many small players have to join the coalition to 'balance' the presence of the large player – in the sense of the input they bring to the coalition. More precisely, m_k is the largest number of small players whose joined input does not outweigh the input of the large player.

The Shapley value for the large player in this case of $(k+1)$ -person game is of the form:

$$\varphi_k^L = \frac{1}{k+1} \left[\frac{m_k(m_k+1)R}{2} + (k-m_k)L \right]$$

and this time it depends on the number of small players k . Because of the discontinuous form of m_k , calculating the limit requires using the three-sequences theorem. The asymptotic Shapley value for the large player in the generalized glove-market game is given by the formula:

$$\varphi^L = \lim_{k \rightarrow \infty} \varphi_k^L = \frac{L}{2R}(2R-L)$$

The vector of Shapley values for the generalised glove-market game is as follows:

$$\varphi = (\varphi^L, \varphi^R) = \left(\frac{L}{2R}(2R-L), \frac{L^2}{2R} \right) \quad (6)$$

□

Remark When $L \geq R$, the type of small players (holder of the smaller amount R) gains $R/2$, when $L < R$, the large player (holder of the smaller amount L) gains $(L/2R)(2R - L) > L/2$, which means that positions of the large player and the type of small players are not symmetric in the generalised glove game.

In the standard n -person glove-market game, switching from the case $L \geq R$ to the case $L < R$ results in exchanging the names and the sequence of coefficients in the vector of Shapley value. This is because the players possessing different kinds of gloves are symmetric – the only difference is in the label of their gloves. The exchange of the numbers of left and right gloves, which amounts to the exchange of names, does not influence any other property of the value.

In the generalised version of the glove-market game, switching the names of gloves between the large player and the type of small players has more profound consequences. The change of the relation of their volumes, as can be seen from the theorem, requires using different formula for calculating the Shapley value. The remark above states that the value for the owner of the smaller amount is not symmetric and the large player is favoured. This implies that the large player and the type of small players do not have symmetric positions in games of this kind.

Relations between the Shapley value and the core Let us recall that in order to investigate the relation between the Shapley value φ , which is a vector, and the core, which is the set of imputations, we needed to introduce a pre-imputation connected to the value, φ_c (see section 2.2). For our case of a 1-person 1-type minimum game, we take $\varphi_c = (x, \xi)$, where $x = \varphi^L$ and $\xi(t) \equiv \varphi^R$, $t \in [0, 1]$. To take into account the differences in the formulae for the value, we investigate two cases. For $L \geq R$, the pair φ_c (compare formula (5)) is individually rational w.r.t. both large and small players, which means that φ_c is an imputation. It can also be verified (by means of Theorem 1) that φ_c belongs to the core. When $L < R$, the pair φ_c (formula (6)) is individually rational w.r.t. large and small players, so it is also an imputation. In this case, however, φ_c does not belong to the core.

We can see that in the generalised glove-market game the relation between the amounts possessed by large and small players is crucial not only for deriving the formulae for the Shapley value, but also for determining whether the value belongs to the core.

4. Maximum game

As a counterpart to the generalised glove-market game we take the maximum game, which has a similar structure, but (as will be shown later) different properties. Generally, the maximum game is defined as a multi-type mixed game where large players are assigned numbers L^1, \dots, L^n , types of small players numbers R^1, \dots, R^n , and the characteristic function is of the form:

$$v(\alpha, \beta) = \max \{ \alpha^1 L^1, \dots, \alpha^n L^n; \beta^1 R^1, \dots, \beta^n R^n \} \quad (7)$$

Similarly to the case of the minimum game, we confine ourselves to investigating the situation where there is only one large player and one type of small players. As an example of the maximum game of such construction, we take an introduction of a governmental health-promotion programme. We consider the government to be the large player and the population of citizens the type of small players. The government prepares a country-wide programme, promoting, for example, a lifestyle that reduces the risk of heart diseases: free medical examinations, TV advertisements, posters, etc. On its part, the society is to some extent health-conscious itself and a fraction of people takes good care of themselves, spending money on healthy food, regular visits to doctors and fitness clubs. Let us note that this model can be easily expanded to many players and many types. Instead of one government, we take local authorities as the large players, and instead of the society in general, we take different social groups as the types of small players.

The amount of money that can be invested by the government is the large player's input L , and the money spent by the fraction of the size β of the citizens is equal to βR . The payoff in this game is the improvement of the state of health in the country, which can be expressed in (public) money saved when less people have to be treated for heart diseases. This payoff is the maximum of the amounts invested by the government and the individuals. To see this, let us have a closer look at the situation. As long as the government does not launch its programme, the increase in social health is only due to the investments made by individuals (small players), that is $v(0; \beta R) = \beta R$. We assume that when the programme starts, it first of all reaches the most health-aware people, who tend to take good care of their health self-dependently. In that case, if the government invests less than the citizens altogether, the increase in the public health will again be only as big as the spending of the health-conscious part of the society; when $L \leq \beta R$, $v(L; \beta R) = \beta R$. At larger cost, the government can reach a greater part of the society, which results in greater improvement of the social health; when $L > \beta R$, $v(L; \beta R) = L$.

In order to calculate the asymptotic Shapley value for the maximum game, we consider the sequence of $(k+1)$ -person games. The coalition in each of these games is denoted:

$C = (a; b^1, \dots, b^k)$, where $a, b^1, \dots, b^k \in \{0, 1\}$, and the characteristic function is given by:

$$v(a, \mathbf{b}) = \max \left\{ aL, \sum_{i=1}^k \frac{R}{k} b^i \right\} \quad (8)$$

In this case, for determining the limit value we have the following:

Theorem 3 Let v be a maximum game with one large player and one type of small players, who possess amounts of L and R , respectively. The asymptotic Shapley value φ of the game v is as follows:

1. $\varphi = \left(L - \frac{R}{2}, \frac{R}{2} \right)$ when $L \geq R$
2. $\varphi = \left(\frac{L^2}{2R}, R - \frac{L^2}{2R} \right)$ when $L < R$

Proof It is analogous to the one for the minimum game (see Theorem 2) and therefore it is omitted here.

Similarly to the situation in the minimum game, when $L \geq R$, the Shapley value for the large player does not depend on the number of small players k . This means that the growing number of small players in the sequence of games does not have an influence on the power (measured by the Shapley value) of the large player – and on the joined power of all small players of one type. So in this case, the value in any game with a large player and a number of small players in one type is equal to the value in a 2-person game.

Remark When $L \geq R$, the type of small players (holder of the smaller amount R) gains $R/2$, when $L < R$, the large player (holder of the smaller amount L) gains $L^2/2R < L/2$, which means that positions of the large player and the type of small players are not symmetric in the maximum game.

As in the minimum game, we observe the lack of symmetry between the large player and the type of small players – exchanging amounts of commodities possessed by them brings a change in the formulae of the Shapley

value and in the allocation of the payoff for the grand coalition. In this game, however, the result $L^2/2R < L/2$ implies that the large player is 'treated less fairly' (as the holder of the smaller amount) than the type of small players, while in the minimum game there was an opposite situation.

Relations between the Shapley value and the core We define a pre-imputation φ_c (i.e. a pair of a number x and a function ξ) connected with the vector φ as the number $x = \varphi^L$ and the function $\xi(t) \equiv \varphi^R$, $t \in [0,1]$.

For the case $L \geq R$, a pre-imputation defined in this way is individually rational w.r.t. small players, but it is not individually rational w.r.t. large players, which means it is not an imputation and as such cannot belong to the core. For the case $L < R$, the pre-imputation φ_c is not individually rational w.r.t. large nor small players, so it is not an imputation and does not belong to the core.

Comparing the above results with analogous results for the minimum game, we see substantial differences in the properties of these games. In the minimum game, the pre-imputation connected to the Shapley value (defined as in section 3 is always an imputation, while in the maximum game the analogous pre-imputation is never an imputation. For the minimum game there is a case ($L \geq R$) when φ_c belongs to the core, while for the maximum game such situation is not possible (at least for φ_c defined as in this section above).

5. Concluding remarks

We present a cooperative multi-type mixed game as an example of games with a finite number of large players and continuum of small ones. The minimum and maximum games are defined in this setup. Calculating the direct formulae for the asymptotic Shapley value enables us to identify the following properties of the multi-type games.

Independence of the number of small players In the investigated games, when $L \geq R$, each $(k+1)$ -person game, $k=1,2,3,\dots$, has the value equal to the value of a 2-person game. Therefore, the sequence of the Shapley values for the type is constant. This means that the type of small players does not 'weaken' when it gains more members, as long as it maintains the possessed amount. This is the case in both the minimum and the maximum games (under the condition that $L \geq R$).

It remains to be verified whether an analogous property, and in what form, holds for the minimum or the maximum games with more large players and

types of small players. It would be useful to identify and describe cases when the type of small players (in theory consisting of infinitely many, and in applications – of a large number of these) enjoys exactly the same power as the type consisting of just one player. In cases when such equivalence (in the sense of the Shapley value) of the type of small players and the large player does not hold, there is a necessity to apply the multi-type mixed games.

Asymmetry of large and small players In the n -person games, players of kinds L and R are symmetric – when their names are exchanged, so are the values for respective kinds of players (with left or right gloves). In both multi-type games investigated here, when the amounts possessed by the large player and the type of small players are exchanged (together with their relation $L \geq R$ or $L < R$), the formula for the asymptotic Shapley value takes a different form. Moreover, the proportion of values for the large player and the type of small players depends not only on the amounts possessed by these agents, but also on the sizes of the players – large and small ones are not treated symmetrically.

This leads to the conclusion that not only is there a substantial difference between large and small players, but also that we cannot treat a type of small players as one more large player. Exchanging the amounts assigned to the large player and to the type of small players results in a non-symmetric change in the proportion of the values.

Relation between the value and the core In the minimum game, changing the relation of the sizes of L and R changes the relation between the value and the core. When $L \geq R$, a pre-imputation connected with the value, φ_c , belongs to the core, while when $L < R$, it does not. In the case of the maximum game, φ_c is not an imputation, so it does not belong to the core.

The core is the set of non-dominated imputations and at the same time, as it is stated by Theorem 1, the set of coalitionally rational imputations. This means that no imputation which belongs the core (allocation of the payoff of the grand coalition) can be vetoed by any coalition. We can think of an element of the core as of the distribution resulting from the negotiations between actors. The Shapley value is something we might call a ‘fair division’, imposed by the impartial arbiter. We see that in the glove game in one case ($L \geq R$) φ_c is a non-dominated imputation and players could agree upon such a distribution, while in the other case ($L < R$) such an agreement would not be possible. In the maximum game, φ_c is not individually rational (w.r.t. large or small players), which means that some players would be better off playing individually, instead of accepting their share of the value. That is the reason why in this game the Shapley value (in spite of its many interesting

properties and an elegant interpretation) would never be the actual distribution of the payoff, if players' opinions were to be considered.

A natural direction for further research would be to find the formulae for asymptotic Shapley value for cases with larger numbers of types and large players. The investigation of different functions ξ^i in the pre-imputations connected with the asymptotic value would show the dependence of the value properties on the form of the function we choose. The examination of other properties (such as monotonicity, convexity, superadditivity or nonemptiness of the core) of the minimum and maximum games will make the picture of these games more complete.

Acknowledgements

The author wishes to thank the referees whose constructive comments helped a lot to prepare this paper to publication.

References

- Aumann, R.J. and L.S. Shapley (1974), *Values of Non-Atomic Games*, Princeton: Princeton University Press.
- Dubey, P. and L.S. Shapley (1979), 'Mathematical properties of the Banzhaf power index', *Mathematics of Operations Research*, 4, 99–131.
- Hart, S. (1973), 'Values of mixed games', *International Journal of Game Theory*, 1, 69–85.
- Hart, S. and D. Monderer (1997), 'Potentials and weighted values of non-atomic games', *Mathematics of Operations Research*, 22, 619–630.
- Kannai, Y. (1966), 'Values of games with a continuum of players', *Israel Journal of Mathematics*, 4, 54–58.
- Sabak, A. and A. Wiczcerek (2003), 'Basic concepts of cooperative games with large and small players', *Reports of Institute of Computer Science*, 957, Polish Academy of Sciences.
- Santos, J.C. and J.M. Zarzuelo (1998), 'Mixing weighted values of non-atomic games', *International Journal of Game Theory*, 27, 331–342.
- Shapley, L.S. (1953), 'A value for n -person games', in: H.W. Kuhn and A.W. Tucker (eds.), *Contributions to the Theory of Games*, II, Princeton: Princeton University Press.