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The Public Good Index when Some Voters are Incompatible

J. M. Alonso-Meijide
Dept. of Statistics and Operations Research, University of Santiago de Compostela, Spain
(eMail: meijide@lugo.usc.es)

B. Casas-Méndez
Dept. of Statistics and Operations Research, University of Santiago de Compostela, Spain
(eMail: eigradu3@usc.es)

Abstract Holler (1982) proposed the Public Good Index as a measure of the power of the players in a simple game. Bergantiños et al. (1993) defined a modification of the Shapley-Shubik index (Shapley and Shubik, 1954) for situations in which some players are incompatible. In this paper, we define a modification of the Public Good Index for these type of situations. We illustrate this index with a real-world example taken from the political field. Furthermore, we provide two characterizations of the new index.

Keywords simple game, power index, Public Good Index, incompatibilities, axiomatic characterization

1. Introduction

In the literature on simple games, a central role is occupied by the study of different power indices. Power is a very difficult concept, but we can understand power of an actor in a voting body as the ability to change an outcome and a power index as a measure of this 'ability'. No consensus has been obtained in order to select a power measure. This is the case of many papers, in which the choice of the power index is not theoretically justified and several power indices are employed.

The main power indices that we can find in the literature are the Shapley-Shubik index (Shapley and Shubik, 1954), the Penrose-Banzhaf-Coleman index (Penrose, 1946; Banzhaf, 1965; Coleman, 1971), the Johnston index (John-

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ston, 1978), the Deegan-Packel index (Deegan and Packel, 1979), and the Public Good Index (Holler, 1982).

In a simple game, there exist a voting body N and a set of winning coalitions W. The coalitions not contained in W are the losing coalitions. The intuitive idea is to think of W as those sets of players that can accomplish their goal in a voting situation. A winning coalition is vulnerable when it has at least one member whose removal would cause the resulting coalition to be a losing coalition. An agent is considered critical when his elimination from a winning coalition turns it into a losing coalition. Holler (1982) proposed that we should consider minimal winning coalitions when it comes to measuring power and the outcome is a public good. A minimal winning coalition is one such that all its members are critical, that is, the winning coalitions considered do not include parties unnecessary for the formation of a winning coalition. In a recent publication in this journal, Braham (2005) postulates that a causal explanation of the coalition outcome necessitates to focus on minimal winning coalitions.

Deegan and Packel (1979) proposed an index under the assumptions that all minimal winning coalitions are equiprobable and all the voters belonging to the same minimal winning coalition should share equally the power. The Public Good Index is determined by the number of minimal winning coalitions containing the voter normalized by the sum of such numbers across all the voters.

Barry (1980) labels the difference between success (i.e., the coalition outcome corresponds to the member's preference) and luck (decisiveness). Holler and Packel (1983) show that the Public Good Index takes into consideration this distinction, and therefore seems to be an adequate way of measuring power. Another index to measuring power builds on minimal winning coalition that we can find in the literature is the Member Bargaining Power (Fishburn and Brams, 1996). Holler (1998) demonstrates that the mathematical properties of the Public Good Index and the Member Bargaining Power are identical.

The choice of an index may well depend on the properties of the situation under analysis. To facilitate this choice, some desirable properties have been introduced in the context of power indices. In this paper, we will mention some of these properties, as well as some characterizations of the Public Good Index according to them.

The equal-likelihood assumption underlies the traditional Public Good Index. In the literature, we can find papers on modifications of the traditional power indices that take into account the possibility that some players may be more likely to act together than others (Owen, 1977 and Owen, 1982); in other cases, the communication among players is limited by the existence of a communication graph (Myerson, 1977 and Owen, 1986). In Myerson

(1977), a modification of the Shapley value (the Myerson value) is defined and characterized. In the characterization of this solution, a property of fairness plays a central role. In a second paper (Myerson, 1980), a property of balanced contributions is used to characterize the Myerson value. In Owen (1986), a modification of the Banzhaf value for situations with communication restricted by a graph is defined. In Alonso-Meijide and Fiestras-Janeiro (2006), this modification is characterized by adapting the properties used by Myerson (1977) and Myerson (1980).

In this paper, we consider situations where some players are incompatible, that is, players that cannot cooperate among them by political reasons. The incompatibilities among players are modelized by a graph, that is, if two players are linked by an arc of the graph, these two players are incompatible. These situations are studied in Carreras (1991), Carreras and Owen (1996), and Bergantiños et al. (1993). In the last paper, a modification of the Shapley value (Shapley, 1953) for these situations is defined and characterized. In the characterization, a property of fairness is used. In Vázquez-Brage (1998) situations with a priori unions and incompatible players are studied.

The main objective of our paper is to define and characterize a modification of the Public Good Index when some players are incompatible. In Section 2, we give the main notions related with simple games, the Public Good Index, and its axiomatic characterizations. In Section 3, we present the games with incompatible players, describe the solution proposed by Bergantiños el al. (1993) and give a new characterization of this solution. In Section 4, we define the modification of the Public Good Index for situations with incompatibilities. We make clear this index using it to analyze the Basque Country Parliament in two different elections. In Section 5, we characterize the new index. Section 6 concludes.

2. Preliminaries

2.1 *Simple games*

A game with Transferable Utility (or TU game) is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of players and v, the characteristic function, is a real function on $2^N = \{S | S \subseteq N\}$ with $v(\varnothing) = 0$. A subset $S \subseteq N$ is called a coalition. We denote by $G(N)$ the set of characteristic functions games with player set N. We will use shorthand notation and write $S \cup i$ for the set $S \cup \{i\}$ and $S \backslash i$ for the set $S \backslash \{i\}$.

A *null player* in a game (N, v) is a player $i \in N$ such that $v(S \cup i) = v(S)$ for all $S \subseteq N \backslash i$. Two players $i, j \in N$ are *symmetric* in a game (N, v) if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \backslash \{i, j\}$.

Given a family of games $H \subseteq G(N)$, a *solution* (or a value) on H is a

function f, which assigns to a game $(N, v) \in H$ a vector

$$(f_1(N, v), \ldots, f_n(N, v)) \in \mathbb{R}^n,$$

where the real number $f_i(N, v)$ is the payoff of the player i in the game (N, v) according to f. It is useful to single out a list of standard desirable properties of solutions.

A solution f is *additive* if $f(N, v + w) = f(N, v) + f(N, w)$ for every $(N, v), (N, w) \in H$ such that $(N, v + w) \in H$, where $(v + w)(S) = v(S) + w(S)$ for every $S \subseteq N$.

A solution f satisfies the *null player* property if $f_i(N, v) = 0$ for every $(N, v) \in H$ and every null player $i \in N$.

A solution f is *symmetric* if $f_i(N, v) = f_j(N, v)$ for every $(N, v) \in H$ and for every pair of symmetric players $i, j \in N$.

A solution f is *efficient* if $\sum_{i \in N} f_i(N, v) = v(N)$ for every $(N, v) \in H$.

The most well-known value for $G(N)$ is the Shapley value (Shapley, 1953). We recall here the definition and the characterization of the Shapley value.

The unique solution f defined on $G(N)$ that satisfies additivity, null player, symmetry, and efficiency is the Shapley value. Given a game $(N, v) \in G(N)$, the Shapley value assigns to each player $i \in N$ the real number:

$$\varphi_i(N, v) = \sum_{S \subseteq N \setminus i} \frac{s!(n - s - 1)!}{n!} [v(S \cup i) - v(S)].$$

A $\{0, 1\}$-*game* is a pair (N, W) where N is the set of players and W is a set of subsets of N satisfying $\varnothing \notin W$. Examples of $\{0, 1\}$-games are control games and Dirac games (Feltkamp, 1995). Also, a $\{0, 1\}$-game can model a decision-making process, that is, the voting body and the decision-making rules. For instance, a *simple game* is a $\{0, 1\}$-game (N, W) satisfying the monotonicity property, i.e.,

$$S \subseteq T \subseteq N \text{ and } S \in W \text{ implies } T \in W.$$

This representation of simple games follows the approach by Peleg and Sudhölter (2003). Intuitively, N is the set of members of a committee and W is the set of coalitions that fully control the involved decision problem. For example, every parliament, every town council, and the UN Security Council are committees.

In a simple game (N, W), a coalition $S \subseteq N$ is *winning* if $S \in W$ and is *losing* if $S \notin W$. A winning coalition $S \subseteq N$ is a *minimal winning* coalition if every proper subset of S is a losing coalition, that is, S is a minimal winning coalition in (N, W) if $S \in W$ and $T \notin W$ for any $T \subset S$.

It has become common practice, to associate a TU game (N, v) to every simple game (N, W). In the following, we identify every simple game with its associated TU game. A $\{0,1\}$-game could be defined as a TU game (N, v) such that $v(S) = 1$ or $v(S) = 0$ for every $S \subseteq N$. A simple game is a $\{0,1\}$-game (N, v) where v is a monotone function, that is, $v(S) \leq v(T)$, for every $S \subseteq T \subseteq N$.

Thus, two important subclasses of TU games are the class of $\{0,1\}$-games and the class of simple games. We denote by $ZO(N)$ (Zero-One games) the set of $\{0,1\}$-games with player set N and by $SI(N)$ (Simple games) the set of simple games with player set N. Note that $SI(N) \subseteq ZO(N) \subseteq G(N)$.

In a simple game (N, v), a coalition $S \subseteq N$ is *winning* if $v(S) = 1$, and S is *losing* if $v(S) = 0$. We denote by $W(v)$ the set of winning coalitions of the game (N, v). A winning coalition $S \subseteq N$ is a *minimal winning* coalition in (N, v) if $v(S) = 1$ and $v(T) = 0$ for any $T \subset S$. We denote by $M(v)$ the set of minimal winning coalitions of the game (N, v) and by $M_i(v)$ the subset of $M(v)$ formed by coalitions $S \subseteq N$ such that $i \in S$.

Given a simple game (N, v), a *swing* for a player $i \in N$ is a coalition $S \subseteq N$ such that $i \in S$, S is a winning coalition, and $S \backslash i$ is a losing one. We denote by $\eta_i(v)$ the set of swings for player $i \in N$. A winning coalition $S \subseteq N$ is a minimal winning coalition if and only if $S \in \eta_i(v)$ for every $i \in S$. For a pair of simple games $(N, v), (N, w) \in H$ and for all $S \subseteq N$, we define $(v \vee w)(S) = \max\{v(S), w(S)\}$, that is, a coalition $S \subseteq N$ is winning in the game $(N, v \vee w)$ if S is winning in at least one of the games (N, v) and (N, w).

2.2 *Power indices*

Given a family of games $H \subseteq SI(N)$, a *power index* on H is a function f, which assigns to a simple game $(N, v) \in H$ a vector

$$(f_1(N, v), \ldots, f_n(N, v)) \in \mathbb{R}^n,$$

where the real number $f_i(N, v)$ is the "power" of the player i in the game (N, v) according to f. The power index of a simple game can be interpreted as a measure of the ability of the different players to turn a losing coalition into a winning one.

The main power indices that we can find in the literature are the Shapley-Shubik, Penrose-Banzhaf-Coleman, Deegan-Packel, Johnston, and Public Good Index. In the next subsection, we consider the Public Good Index.

2.3 *Public Good Index*

For the Public Good Index, introduced in Holler (1982), only minimal winning coalitions are considered relevant when it comes to measuring power.

Definition 1 Given a simple game (N, v), *the Public Good Index* assigns to each player $i \in N$ the real number[1]:

$$\delta_i(N, v) = \frac{|M_i(v)|}{\sum_{j \in N} |M_j(v)|}. \tag{1}$$

The Public Good Index of a player i is equal to the total number of minimal winning coalitions containing player i divided by the sum of these number for all players.

An axiomatic characterization of this index for the family of simple games can be found in Holler and Packel (1983), using the properties of efficiency, null player, symmetry, and mergeability. We remember here the property of mergeability.

Two simple games (N, v) and (N, w) are *mergeable* if for all pair of coalitions $S \in M(v)$ and $T \in M(w)$, it holds that $S \nsubseteq T$ and $T \nsubseteq S$. The minimal winning coalitions in game $(N, v \vee w)$ are precisely the union of the minimal winning coalitions in games (N, v) and (N, w). If two games (N, v) and (N, w) are mergeable, the mergeability condition guarantees that $|M(v \vee w)| = |M(v)| + |M(w)|$.

A power index f on $H \subseteq SI(N)$ satisfies the *mergeability* property if for any pair of mergeable simple games $(N, v), (N, w) \in H$ such that $(N, v \vee w) \in H$, it holds that for all player $i \in N$:

$$f_i(N, v \vee w) = \frac{f_i(N, v) \sum_{j \in N} |M_j(v)| + f_i(N, w) \sum_{j \in N} |M_j(w)|}{\sum_{j \in N} |M_j(v \vee w)|}.$$

This property says that if two simple games (N, v) and (N, w) are mergeable, then the power of player i in the game $(N, v \vee w)$ is the sum of his power in games (N, v) and (N, w) (first, this power must be normalized by the number of minimal winning coalitions of every player in games (N, v), (N, w), and $(N, v \vee w)$).

> The unique power index f defined on $SI(N)$ satisfying mergeability, null player, symmetry, and efficiency is the Public Good Index (Holler and Packel, 1983).

In Alonso-Meijide et al. (2007) a new characterization of the Public Good Index is provided, using a property similar to strong monotonicity (Young, 1985) instead of mergeability. This property is named minimal monotonicity. It takes into account a relation between two simple games (N, v) and (N, w),

[1] In the trivial game (N, v) with $v(S) = 0$ for every $S \subseteq N$, we define $\delta_i(N, v) = 0$ for every $i \in N$. We denote by $|S|$ the cardinality of a set S.

that is, given in terms of the cardinality of the sets of minimal winning coalitions.

A power index f on $H \subseteq SI(N)$ satisfies the property of *minimal monotonicity* if for any pair of games (N, v), $(N, w) \in H$, it holds that:

$$f_i(N, w) \sum_{j \in N} \left|M_j(w)\right| \geq f_i(N, v) \sum_{j \in N} \left|M_j(v)\right|,$$

for all player $i \in N$ such that $M_i(v) \subseteq M_i(w)$.

This property states that if the set of minimal winning coalitions containing a player i in game (N, v) is a subset of minimal winning coalitions containing this player in game (N, w), then the power of player i in game (N, w) is not less than power of player i in game (N, v) (first, this power must be normalized by the number of minimal winning coalitions of every player in games (N, v) and (N, w)).

The unique power index f on $SI(N)$ satisfying minimal monotonicity, null player, symmetry, and efficiency is the Public Good Index (Alonso-Meijide et al., 2007).

3. Games with incompatibilities

Let $N = \{1, 2, \ldots, n\}$ be a finite set. An undirected graph without loops on N is a set of unordered pairs of different elements. These pairs $i : j$ are called links. (Note that $i : j = j : i$).

We denote by g^N the complete graph on N and by $GR\,(N)$ the set of undirected graphs on N, that is:

$$g^N = \{i : j \mid i \in N, j \in N, i \neq j\} \text{ and } GR(N) = \{g | g \subseteq g^N\}.$$

Let us take $g \in GR\,(N)$. Given a pair of agents $i, j \in N$, we say that i and j are incompatible if $i : j \in g$. We say that two players i and j are incompatible if they cannot cooperate at all between them. We say that a coalition S is g-admissible if and only if $i : j \notin g$ for all $i, j \in S$. The case $g = \varnothing$, represents the situation in which no incompatible players exist.

If $g \in GR(N)$, for any coalition $S \subseteq N$, we denote by $P(S, g)$ the set of all partitions of S whose classes are g–admissible coalitions. Given $g \in GR(N)$, g^* denotes the dual graph of g, i.e., the element of $GR(N)$ $g^* = \{i : j \mid i, j \in N,\ i \neq j,\ i : j \notin g\}$. For any $S \subseteq N$ and any $g \in GR(N)$, we say that $i, j \in S$ are connected in S by g if g induces a path in S connecting i and j, and we denote by S/g the set of connected components of S (maximal subsets of connected elements of S) associated to the graph g. Note that S/g is a partition of S.

Given $i : j \in g$, the graph $g \backslash i : j \in GR\,(N)$ is defined as the resulting graph after the elimination of the link $i : j$, that is:

$$g \backslash i : j = \{h : k \in g \mid h : k \neq i : j\}.$$

For any $i \in N$, we denote by g_{-i} the element of $GR(N)$ obtained by breaking the links where agent i is involved, i.e.,

$$g_{-i} = \{j : h \in g \mid j \neq i \text{ and } h \neq i\}.$$

A situation with incompatibilities is a triplet (N, v, g) where (N, v) is a TU game and $g \in GR(N)$. We denote by $I(N)$ the set of TU games with incompatibilities with player set N, and by $C(N)$ the set of simple games with incompatibilities with player set N.

In Bergantiños et al. (1993), an efficient and fair allocation rule defined on $I(N)$ is introduced, which selects a payoff for every possible situation with incompatibilities, and they prove that it is uniquely determined. Adapting the ideas of Myerson (1977) to this context, they consider the desirable following properties for an allocation rule f defined on $I(N)$.

An allocation rule satisfies *i-component efficiency* if

$$\sum_{i \in S} f_i(N, v, g) = \max_{P \in P(S,g)} \sum_{T \in P} v(T), \tag{2}$$

for every $(N, v, g) \in I(N)$ and every $S \in N/g^*$.

With respect to this property observe the following. N/g^* partitions N in such a way that, if i and j are in different classes of the partition, then i and j are incompatible. Hence, it is clear that these $S \in N/g^*$ are the coalitional unities within which players can negotiate and form systems of admissible coalitions; it seems logical to suppose that they will form them in the best possible way, that is in such a way that the sum of the payoffs obtained by the players in every $S \in N/g^*$ is the biggest possible. This is simply what i-component efficiency asks for. Observe that, in general, if $S \in N/g^*$, then

$$\max_{P \in P(S,g)} \sum_{T \in P} v(T) \neq v(S),$$

because, although $S \in N/g^*$, maybe it is not g-admissible. Hence, this efficiency condition of allocation rules for situations with incompatibilities is not a kind of dual of Myerson's efficiency of allocation rules for communication situations (see Myerson, 1977).

An allocation rule satisfies *i-fairness* if

$$f_i(N, v, g) - f_i(N, v, g \cup i : j) = f_j(N, v, g) - f_j(N, v, g \cup i : j),$$

for every $(N, v, g) \in I(N)$ and every $i : j \in g^*$.

The i-fairness condition is a property that reflects the symmetry of the binary relation underlying to the graph: if two players become compatible, then they win or lose the same.

An allocation rule satisfies *i-stability* if

$$f_i(N, v, g) \geq f_i(N, v, g \cup i : j) \text{ and } f_j(N, v, g) \geq f_j(N, v, g \cup i : j),$$

for every $(N, v, g) \in I(N)$ and every $i : j \in g^*$.

The i-stability property expresses that for a player, being incompatible with others can only damage him.

Bergantiños et al. (1993) define the g-restriction of a game in a situation with incompatibilities.

Definition 2 *The g-restriction of v*, given a situation with incompatibilities (N, v, g), is the TU-game (N, v^g) given by:

$$v^g(S) = \sum_{T \in S/g^*} \max_{P \in P(T,g)} \sum_{U \in P} v(U) = \max_{P \in P(S,g)} \sum_{U \in P} v(U) \text{ for every } S \subseteq N.$$

This definition has the same flavour than the property of i-component efficiency. The players of a coalition S form g-admissible subcoalitions (which are the only possible coalitions) and they select the system of subcoalitions that maximize the total payoffs of the players in S. The g-restricted game is always superadditive. Bergantiños et al. (1993) use the g-restricted game and the following theorem to propose their allocation rule, related with the Shapley value.

Theorem 3 (Bergantiños et al. (1993)) There exists a unique allocation rule $\mathbf{f}$ on $I(N)$ satisfying i-component efficiency and i-fairness. This allocation rule is given by:

$$\mathbf{f}(N, v, g) = \varphi(N, v^g) \text{ for every } (N, v, g) \in I(N).$$

Moreover, this allocation rule satisfies i-stability and for every TU-game (N, v), $\mathbf{f}(N, v, \varnothing) = \varphi(N, v)$.

One important axiom in the literature is balanced contributions. This is based on a principle of reciprocity, as introduced by Myerson (1980), and is often used in the literature on the Shapley value. Myerson's property of balanced contributions asserts that for any two players the gain or loss to each player when the other "leaves" the game should be equal.

An allocation rule satisfies *i-balanced contributions* if

$$f_i(N, v, g) - f_i(N, v, g_{*j}) = f_j(N, v, g) - f_j(N, v, g_{*i}),$$

for every $(N, v, g) \in I(N)$ and every $i, j \in N$, where $g_{*i} = g \cup \{i : k \in g^* \mid k \in N\}$ is the graph obtained from g if player i becomes incompatible with all the players.

This property says that the loss or gain for player i when player j becomes incompatible with all the players is the same as the loss or gain for player j when player i becomes incompatible with all the players. We obtain a new characterization.

Theorem 4 There exists a unique allocation rule $\mathbf{f}$ on $I(N)$ satisfying i-component efficiency and i-balanced contributions. This allocation rule is given by:

$$\mathbf{f}(N, v, g) = \varphi(N, v^g) \text{ for every } (N, v, g) \in I(N).$$

(Sketch of the proof) In Vázquez-Brage (1998), the Owen value (1977) when some players are incompatible is introduced. There, it is proved that this modification of the Owen value satisfies a property of balanced contributions. By following similar arguments, we can prove that the allocation rule of the theorem satisfies *i-balanced contributions*. Taking into account that an allocation rule that satisfies *i-balanced contributions* also satisfies *i-fairness*, by Theorem 3, the result is proved.

4. A modification of the PGI for simple games with incompatibilities

According to the interpretation of the Public Good Index, in a wide class of situations, the power of a player should be based on his participation in the formation of minimal winning coalitions. Taking into account this fact, in this section we introduce an allocation rule for simple games with incompatibilities, i.e., a map $f : C(N) \longrightarrow \mathbb{R}^n$ which allocates a vector of payoffs (to the players) to any situation with incompatibilities in $C(N)$.

Definition 5 *The Public Good Index with incompatibilities* is the allocation rule that assigns to every situation with incompatibilities $(N, v, g) \in C(N)$ the vector of payoffs $\delta(N, v^g)$ where δ is the Public Good Index and (N, v^g) the g-restriction of v.

We present two examples that illustrate the index that we propose for simple games when some voters are incompatible. In the examples we analyze the Basque Country Parliament, one of the seventeen Spanish autonomous communities. We will employ two different elections, the first one, held on November 1986, and the second one, held on April 2005.

Example 6 *The Basque Country Parliament, 1986*
The Basque Country Parliament is made up of 75 members. We use the parties involved after the elections held on 30 November 1986 with the objective of comparing our results with those obtained by Carreras and Owen (1996), using a modification of the Shapley-Shubik index.

Following the elections held on 30 November 1986, seven parties are involved in the final results. We designate each party in the text by its initials: PSE (19 seats), PNV (17 seats), EA (13 seats), HB (13 seats), EE (9 seats), CP (2 seats), and CDS (2 seats). PSE is a socialist party, PNV, EA, HB, and EE are

Table 1 — Shapley-Shubik Index and Public Good Index

Party	Seats	φ	δ
PSE	25.33%	0.2524	0.2105
PNV	22.67%	0.2524	0.2105
EA	17.33%	0.1524	0.1579
HB	17.33%	0.1524	0.1579
EE	12.00%	0.1524	0.1579
CP	2.67%	0.0190	0.0526
CDS	2.67%	0.0190	0.0526

nationalist, and CP and CDS are center-right and center parties, respectively. For more details about the description of these parties and their political positions the reader is referenced to Carreras and Owen (1996).

The total of seats is 75 and then, a majority of 38 is required to obtain a winning coalition. The set of minimal winning coalitions is formed by 12 elements:

PSE, PNV, EA	PSE, EA, EE
PSE, PNV, HB	PSE, HB, EE
PSE, PNV, EE	PNV, EA, HB
PSE, PNV, CP	PNV, EA, EE
PSE, PNV, CDS	PNV, HB, EE
PSE, EA, HB	EA, HB, EE, CP, CDS.

In the Table 1, we include the percentages of seats, the Shapley-Shubik index, and the Public Good Index for each party involved in our analysis.

We introduce here some information about no feasible coalitions taking into account the previous behavior and the opinions expressed by the leaders of each party. We consider a graph of incompatibilities formed by 5 links:

PSE:HB PSE:CP HB:EE HB:CP HB:CDS.

Taking into account these incompatibilities, the cardinality of the set of admissible minimal winning coalitions is 6:

PSE, PNV, EA	PSE, EA, EE
PSE, PNV, EE	PNV, EA, HB
PSE, PNV, CDS	PNV, EA, EE.

In Table 2, we present the Shapley-Shubik index and the Public Good Index taking into account the graph of incompatibilities.

Table 2 — Shapley-Shubik Index and
Public Good Index with incompatibilities

Party	Seats	φ	δ
PSE	25.33%	0.2333	0.2222
PNV	22.67%	0.3167	0.2778
EA	17.33%	0.2333	0.2222
HB	17.33%	0.0333	0.0555
EE	12.00%	0.1500	0.1667
CP	2.67%	0.0000	0.0000
CDS	2.67%	0.3333	0.0555

The results are similar with both indices, that is, PNV and EA increase their power, PNV ranks first, even though PSE has more seats. Taking into account the rejections, it is expected the regression of HB. Now, CP is a null player because of its rejection to PSE and HB.

Example 7 The Basque Country Parliament, 2005
According to the elections of April, 2005, the Parliament was composed of 29 members of PNV-EA (party 1), 18 members of PSE-PSOE (party 2), 15 members of PP (party 3), 3 members of the communist party IU-EB (party 5), and 9 and 1 members, respectively, of the left-wing regionalist parties PCTV (party 4), and ARALAR (party 6). In this case, the set of minimal winning coalitions is formed by 4 coalitions:

PNV-EA, PSE-PSOE PNV-EA, PP
PNV-EA, PCTV PSE-PSOE, PP, PCTV.

If we compute the PGI, we obtain the next vector:

$$\delta(N, v) = (\frac{1}{3}, \frac{2}{9}, \frac{2}{9}, \frac{2}{9}, 0, 0).$$

As we did for the elections held on 1986, we introduce here some information about infeasible coalitions taking into account the previous behavior and the opinions expressed by the leaders of each party. We consider a graph of incompatibilities formed by 3 links:

PNV-EA:PP PP:PCTV PSE-PSOE:PCTV.

We might perhaps have considered other rejections, but this would have had no effect on our current list of admissible minimal winning coalitions.

Taking into account the considered incompatibilities, the set of minimal winning coalitions is formed by two coalitions

PNV-EA, PSE-PSOE PNV-EA, PCTV.

In this case, if we compute the PGI, we obtain the next vector:

$$\delta(N,v) = (\frac{1}{2}, \frac{1}{4}, 0, \frac{1}{4}, 0, 0).$$

5. Axiomatic characterizations

We introduce new properties in the context of simple games with incompatibilities. These properties are related with those proposed by Bergantiños et al. (1993) introducing in its formulation the set of minimal winning coalitions of the corresponding g-restricted games taking into account the fact that in a wide class of situations, the power of a player should be based on his participation in the formation of minimal winning coalitions.

An allocation rule satisfies *minimal i-fairness* if

$$\sum_{k\in N} \mid M_k(v^g) \mid [f_i(N,v,g) - f_j(N,v,g)]$$

$$= \sum_{k\in N} \mid M_k(v^{g\cup i:j}) \mid [f_i(N,v,g\cup i:j) - f_j(N,v,g\cup i:j)], \tag{3}$$

for every $(N,v,g) \in C(N)$ and every $i:j \in g^*$.

Minimal i-fairness states that the difference of power between two players does not change if the players become incompatibles (previously, we must normalize this power by the total number of minimal winning coalitions in which the players are involved in both restricted games), that is, if two players become incompatible this fact affects in the same way to the players, after the normalization of the games.

An allocation rule satisfies *minimal i-stability* if

$$\sum_{k\in N} \mid M_k(v^g) \mid f_i(N,v,g) \geq \sum_{k\in N} \mid M_k(v^{g\cup i:j}) \mid f_i(N,v,g\cup i:j) \text{ and}$$

$$\sum_{k\in N} \mid M_k(v^g) \mid f_j(N,v,g) \geq \sum_{k\in N} \mid M_k(v^{g\cup i:j}) \mid f_j(N,v,g\cup i:j), \tag{4}$$

for every $(N,v,g) \in C(N)$ and every $i:j \in g^*$.

The minimal i-stability property expresses that for a player, being incompatible with others can only damage him (previously, we must normalize the power of a player in each considered game with incompatibilities by the total

number of minimal winning coalitions in which the players are involved in the corresponding restricted games).

An allocation rule satisfies *minimal i-balanced contributions* if

$$\sum_{k\in N} \mid M_k(v^g) \mid f_i(N,v,g) - \sum_{k\in N} \mid M_k(v^{g_{*j}}) \mid f_i(N,v,g_{*j})$$

$$= \sum_{k\in N} \mid M_k(v^g) \mid f_j(N,v,g) - \sum_{k\in N} \mid M_k(v^{g_{*i}}) \mid f_j(N,v,g_{*i}) \tag{5}$$

for every $(N,v,g) \in C(N)$ and every $i, j \in N$.

This property says that the loss or gain for player i when player j becomes incompatible with all the players is the same as the loss or gain for player j when player i becomes incompatible with all the players (previously, we must normalize the power of a player in each considered game with incompatibilities by the total number of minimal winning coalitions in which the players are involved in the corresponding restricted games).

An allocation rule satisfies *null game* if $f_i(N,v,g) = 0$ for every $(N,v,g) \in C(N)$ such that $M(v^g) = \varnothing$ and every $i \in N$. The property of null game is a natural request.

We propose our first characterization result. We point up that the line of the proof is different to the corresponding one in the work of Bergantiños et al. (1993) and the main reason is that in this setup we can not use the property of additivity.

Theorem 8 There exists a unique allocation rule $\mathbf{f}$ on $C(N)$ satisfying null game, i-component efficiency, and minimal i-fairness. This allocation rule is given by:

$$\mathbf{f}(N,v,g) = \delta(N,v^g) \text{ for every } (N,v,g) \in C(N).$$

Moreover, this allocation rule satisfies minimal i-stability and for every simple game (N,v), $\mathbf{f}(N,v,\varnothing) = \delta(N,v)$.

Proof We prove the first part of the theorem. *Existence.* It suffices to show that the allocation rule given by $f(N,v,g) = \delta(N,v^g)$ for every $(N,v,g) \in C(N)$ satisfies *null game*, *i-component efficiency*, and *minimal i-fairness*. This allocation rule satisfies *null game*, because if $v^g = 0$ then $\delta_i(N,v^g) = 0$ for every $i \in N$.

If $v^g = 0$, it satisfies *i-component efficiency* (2) because *null game* implies that $\delta_i(N,v^g) = 0$ for every $i \in N$. Hence, for every $S \in N/g^*$, $\sum_{i\in S} \delta_i(N,v^g) = 0$.

Suppose that $v^g \neq 0$, then there is $S \in N/g^*$ such that S is a winning coalition. Let $\{S_1, \cdots, S_t\}$ such that $i \in I = \{1, \cdots, t\}$ if and only if $S_i \in M(v)$

and S_i g-admissible. Then, $M(v^g) = \{S_1, \cdots, S_t\}$ and $v^g = u_{S_1} \vee \cdots \vee u_{S_t}$[2], because for every $i \in I$, $v^g(S_i) = v(S_i) = 1$ and if $T \subset S_i$ then $v^g(T) = 0$. As the games $u_{S_1}, \cdots, u_{S_t}$ are mergeable and the Public Good Index satisfies a property of *mergeability* we have that:

$$\sum_{k \in N} \mid M_k(v^g) \mid \delta_i(N, v^g) = \sum_{j \in I} \mid S_j \mid \delta_i(N, u_{S_j}) = \sum_{j \in I \mid i \in S_j} 1 \; \forall i \in S,$$

where the last equality is a consequence of that i is a null player in (N, u_{S_j}) if $i \notin S_j$. Hence,

$$\sum_{i \in S} \delta_i(N, v^g) = \frac{1}{\sum_{k \in N} \mid M_k(v^g) \mid} \sum_{i \in S} \sum_{j \in I \mid i \in S_j} 1 = 1,$$

and the allocation rule given by $f(N, v, g) = \delta(N, v^g)$ satisfies *i-component efficiency*.

Now, we prove that this allocation rule satisfies *minimal i-fairness* introduced in expression (3).

Take $(N, v, g) \in C(N)$ and $i : j \in g^*$, then

$$\sum_{k=1}^{n} \mid M_k(v^{g \cup i:j}) \mid [\delta_i(N, v, g \cup i : j) - \delta_j(N, v, g \cup i : j)]$$

$$= \mid M_i(v^{g \cup i:j}) \mid - \mid M_j(v^{g \cup i:j}) \mid$$

$$= \mid M_i(v^{g \cup i:j}) \backslash M_j(v^{g \cup i:j}) \mid - \mid M_j(v^{g \cup i:j}) \backslash M_i(v^{g \cup i:j}) \mid . \tag{6}$$

If $S \in M_i(v^{g \cup i:j}) \backslash M_j(v^{g \cup i:j})$, S is minimal winning in $(N, v^{g \cup i:j})$ in such way that S is $g \cup i : j$-admissible. Moreover, $i \in S$ and $j \notin S$. Hence, S is g-admissible with the graph of incompatibilities g and minimal winning in (N, v^g), that is, $S \in M_i(v^g) \backslash M_j(v^g)$. With a similar reasoning, if $S \in M_i(v^g) \backslash M_j(v^g)$ then $S \in M_i(v^{g \cup i:j}) \backslash M_j(v^{g \cup i:j})$. If $M_i(v^{g \cup i:j}) \backslash M_j(v^{g \cup i:j}) = \varnothing$, then $M_i(v^g) \backslash M_j(v^g) = \varnothing$, and for all cases

$$M_i(v^g) \backslash M_j(v^g) = M_i(v^{g \cup i:j}) \backslash M_j(v^{g \cup i:j}),$$

and in a similar way,

$$M_j(v^g) \backslash M_i(v^g) = M_j(v^{g \cup i:j}) \backslash M_i(v^{g \cup i:j}).$$

As a consequence of this, the expression (6) is equal to

$$\mid M_i(v^g) \backslash M_j(v^g) \mid - \mid M_j(v^g) \backslash M_i(v^g) \mid$$

[2] If $S \subseteq N$, (N, u_S) denotes the unanimity game defined as $u_S(T) = 1$ if $S \subseteq T$ and $u_S(T) = 0$ in other case.

$$= \sum_{k=1}^{n} \mid M_k(v^g) \mid [\delta_i(N,v,g) - \delta_j(N,v,g)].$$

Uniqueness. Suppose that there exist two solutions f^1 and f^2 that satisfy *null game*, *i-component efficiency*, and *minimal i-fairness*. We can find a proper simple game (N,v) and a graph of incompatibilities $g \in GR(N)$ with the maximal number of links such that $f^1(N,v,g) \neq f^2(N,v,g)$.

If $g^* = \varnothing$ (i. e. $g = g^N$), then the connected components of N/g^* are the players and by *i-component efficiency* we obtain that $f_i^1(N,v,g) = f_i^2(N,v,g)$ for every $i \in N$.

If $g^* \neq \varnothing$ (i. e. $g \neq g^N$) and $\mid M(v^g) \mid = 0$, by the property of *null game* we obtain $f_i^1(N,v,g) = f_i^2(N,v,g) = 0$ for every $i \in N$.

Finally, suppose that $g^* \neq \varnothing$ and $\mid M(v^g) \mid > 0$. Let $i \in N$, we consider two cases:

1. $\{i\} \in N/g^*$. Then $f_i^1(N,v,g) = f_i^2(N,v,g)$ by the property of *i-component efficiency*.

2. $\exists j \in N$ such that $i : j \in g^*$. By the maximality of g, $f_k^1(N,v,g \cup i : j) = f_k^2(N,v,g \cup i : j)$ for every $k \in N$. Taking into account the property of *minimal i-fairness* (3), we obtain

$$\sum_{k=1}^{n} \mid M_k(v^g) \mid (f_i^1(N,v,g) - f_j^1(N,v,g))$$

$$= \sum_{k=1}^{n} \mid M_k(v^{g \cup i:j}) \mid (f_i^1(N,v,g \cup i : j) - f_j^1(N,v,g \cup i : j))$$

$$= \sum_{k=1}^{n} \mid M_k(v^{g \cup i:j}) \mid (f_i^2(N,v,g \cup i : j) - f_j^2(N,v,g \cup i : j))$$

$$= \sum_{k=1}^{n} \mid M_k(v^g) \mid (f_i^2(N,v,g) - f_j^2(N,v,g)).$$

Hence, we have

$$f_i^1(N,v,g) - f_i^2(N,v,g) = f_j^1(N,v,g) - f_j^2(N,v,g) \tag{7}$$

for every couple of players i, j such that $i : j \in g^*$. Moreover, given $S \in N/g^*$ and every couple of players $i, j \in S$, we also have the expression (7), and then $f_i^1(N,v,g) - f_i^2(N,v,g) = d_S(g)$ for each $i \in S$ where $d_S(g)$ depends only on S and g but it no depends on i. By the property of *i-component efficiency*, we have

$$\sum_{i \in S} f_i^1(N,v,g) = \sum_{i \in S} f_i^2(N,v,g),$$

then $0 = \sum_{i \in S}(f_i^1(N, v, g) - f_i^2(N, v, g)) = \mid S \mid d_S(g)$, and so $d_S(g) = 0$ for every $S \in N/g^*$. Then, we have $f_i^1(N, v, g) = f_i^2(N, v, g)$ for every $i \in N$.

Now, we prove that our allocation rule satisfies *minimal i-stability* given in expression (4). Taking into account that the Public Good Index satisfies *minimal monotonicity*, to prove the result it is enough to see that if $(N, v, g) \in C(N)$ for every $i : j \in g^*$ we have that $M_i(v^{g \cup i:j}) \subset M_i(v^g)$.

If $S \in M_i(v^{g \cup i:j})$, S is a minimal winning coalition that is admissible by $g \cup i : j$. In the same way, we have that $S \in M_i(v^g)$ because S is also admissible by g. The proof of the result for player j is the same.

Finally, the last statement of the theorem follows immediately from the definition of the Public Good Index when some players are incompatible. □

Lemma 9 If a power index $f : C(N) \longrightarrow \mathbb{R}^n$ satisfies minimal i-balanced contributions then it satisfies minimal i-fairness.

Proof Note that if $g \in GR(N)$, $i : j \in g^*$, and $g' = g \cup i : j$ then $g'_{*i} = g_{*i}$ and $g'_{*j} = g_{*j}$.

Because f satisfies *minimal i-balanced contributions* (5), if $(N, v, g) \in C(N)$, for every $i : j \in g^*$ we have:

$$\sum_{k \in N} \mid M_k(v^g) \mid f_i(N, v, g) - \sum_{k \in N} \mid M_k(v^{g \cup i:j}) \mid f_i(N, v, g \cup i : j)$$

$$= \sum_{k \in N} \mid M_k(v^{g_{*j}}) \mid f_i(N, v, g_{*j}) + \sum_{k \in N} \mid M_k(v^g) \mid f_j(N, v, g)$$

$$- \sum_{k \in N} \mid M_k(v^{g_{*i}}) \mid f_j(N, v, g_{*i}) - \sum_{k \in N} \mid M_k(v^{g_{*j}}) \mid f_i(N, v, g_{*j})$$

$$- \sum_{k \in N} \mid M_k(v^{g \cup i:j}) \mid f_j(N, v, g \cup i : j) + \sum_{k \in N} \mid M_k(v^{g_{*i}}) \mid f_j(N, v, g_{*i})$$

$$= \sum_{k \in N} \mid M_k(v^g) \mid f_j(N, v, g) - \sum_{k \in N} \mid M_k(v^{g \cup i:j}) \mid f_j(N, v, g \cup i : j)$$

and f satisfies *minimal i-fairness.* □

Theorem 10 There exists a unique allocation rule $\mathbf{f}$ on $C(N)$ satisfying i-component efficiency and minimal i-balanced contributions. This allocation rule is given by

$$\mathbf{f}(N, v, g) = \delta(N, v^g) \text{ for every } (N, v, g) \in C(N).$$

Proof It is enough to prove that the power index $\delta(N, v^g)$ satisfies *minimal i-balanced contributions*. For $(N, v, g) \in C(N)$, if $\mid M(v^g) \mid = 0$, the result is trivial. If $\mid M(v^g) \mid > 0$, for every couple of players $i, j \in N$,

$$\sum_{k \in N} \mid M_k(v^g) \mid \delta_i(N, v^g) - \sum_{k \in N} \mid M_k(v^{g_{*j}}) \mid \delta_i(N, v^{g_{*j}})$$

$$= \mid M_i(v^g) \mid - \mid M_i(v^{g_{*j}}) \mid$$

$$= \mid M_i(v^g) \backslash M_j(v^g) \mid + \mid M_i(v^g) \cap M_j(v^g) \mid$$

$$- \mid M_i(v^{g_{*j}}) \backslash M_j(v^{g_{*j}}) \mid - \mid M_i(v^{g_{*j}}) \cap M_j(v^{g_{*j}}) \mid . \tag{8}$$

It is easy to see that $M_i(v^g) \backslash M_j(v^g) = M_i(v^{g_{*j}}) \backslash M_j(v^{g_{*j}})$ and $M_i(v^{g_{*j}}) \cap M_j(v^{g_{*j}}) = M_j(v^{g_{*i}}) \cap M_i(v^{g_{*i}}) = \varnothing$. Then, expression (8) depends on i in the same way that it depends on j, and hence it is equal to

$$\mid M_j(v^g) \mid - \mid M_j(v^{g_{*i}}) \mid$$

$$= \sum_{k \in N} \mid M_k(v^g) \mid \delta_j(N, v^g) - \sum_{k \in N} \mid M_k(v^{g_{*i}}) \mid \delta_j(N, v^{g_{*i}}). \qquad \square$$

6. Concluding remarks

In this paper, we define and characterize a modification of the Public Good Index when some pair of players, corresponding to the links of a certain graph g, are not admissible. The properties that characterize the proposed solution are similar to those used by Bergantiños et al. (1993) to characterize a modification of the Shapley value in the same context.

We illustrate our definition with two examples in which we include a brief analysis of the Basque Country Parliament. The index we propose has given us a criterion of stability, allowing us to select some minimal winning coalitions as the most likely ones to form. This index could be used to measure power in similar situations.

As a referee has pointed out, it would be of interest to analyze as the modifications of the graph, i.e., adding (or removing) a certain link, may change the voting power. It is though natural to ask a question how to quantify these changes. For each existing graph one may distinguish the "crucial link" such its deletion introduces the largest changes of the voting power. Furthermore, it would be interesting to analyze a similar question from the point of view of a given player i : which links added (or removed) to the graph from the vertex i imply a largest shift of the voting power? For example, in the Basque Country Parliament, 2005 , if we remove the link PNV-EA:PP, the modified PGI index of PNV-EA is the same $(1/2)$, meanwhile that PP receives a power of $1/6$. This power is lost by PSE-PSOE and PCTV (1/12 each one).

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