

A Geometric Approach to Paradoxes of Majority Voting: From Anscombe's Paradox to the Discursive Dilemma with Saari and Nurmi

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Abstract Among many other topics, Hannu Nurmi has worked on voting paradoxes and how to deal with them. In his work he often uses a geometric approach developed by Don Saari for the analysis of paradoxes of preference aggregation such as the Condorcet paradox or Arrow's general possibility theorem. In this paper we extend his approach to other paradoxes analysed by Nurmi and the recent work in judgment aggregation. In particular we use Saari's representation cubes to provide a geometric representation of profiles and majority outcomes. Within this geometric framework, we show how profile decompositions can be used to derive restrictions on profiles that avoid the paradoxes of majority voting. Moreover, we use our framework to determine the likelihood of those paradoxes. Finally, current distance-based approaches in judgment aggregation are discussed within our framework.

Keywords judgment aggregation, voting paradoxes, Arrow's theorem

1. Introduction

In the last thirty years, there have been several attempts to generalize the Arrowian framework of preference aggregation (e.g. Rubinstein and Fishburn,

1986, or Wilson, 1975). This literature on abstract aggregation has been considerably stimulated by the growing interest in problems of judgment aggregation. The problem of judgment aggregation consists in aggregating individual judgments on an agenda of logically interconnected propositions into a collective set of judgments on these propositions (see List and Puppe, 2007, for a survey).

As an example of a paradox in judgment aggregation, consider a variant of the so-called discursive dilemma, in which a committee of three recruitment officers in a firm has to decide whether a job applicant should be hired or not. There is a written test and an oral interview and each of them is advised to recommend hiring the applicant if and only if the applicant passes the written test and gives a satisfiable interview. Table 1.1 shows the judgments of the officers and their majority decisions.

Officer	written test	oral interview	decision
Officer 1	1	1	1
Officer 2	1	0	0
Officer 3	0	1	0
Majority Outcome	1	1	0

Table 1.1 — Discursive Dilemma

Based on the majority of the individual decisions, the job applicant will not be hired as a majority does not find her acceptable. However, a majority finds the written test as well as the interview acceptable.

Problems of judgment aggregation are structurally similar to paradoxes and problems in social choice theory like the Condorcet paradox and Arrow's general possibility theorem, but also related to paradoxes of compound majorities like the Anscombe or Ostrogorski paradoxes, both nicely analysed by Nurmi (1997) (see also Nurmi, 1987 and 1999). Contemporarily, Saari (1995) has developed and popularized a geometric approach to Arrovian social choice theory. His approach has helped to understand what drives many of the impossibility results and paradoxes in social choice theory.

In a similar vein falls Nurmi's (2004) distance-based approach to aggregation problems, something that can be seen as one of the earliest attempts to apply a geometric approach to abstract aggregation problems. E.g. Meskanen and Nurmi (2006) show that all of the aggregation rules devised to overcome the problem of majority cycles (e.g. Copeland rule, Kemeny rule) can be characterized by a distance measure and a certain goal state.

In this paper we develop - in Saari's style - a geometric approach to ab-

stract aggregation theory starting from a paradox intensively investigated by Nurmi and extending this framework to typical paradoxes in judgment aggregation.

Our approach focuses on and will be exhaustive for aggregation problems that can be represented in the three-dimensional hypercube. While this is the smallest dimension in which interesting aggregation problems can be formulated and be particularly illuminating for problems that naturally fall into this framework, we have to give a warning that most of our results are not easily extendable to more than three dimensions.

A major difference of judgment aggregation to social choice theory lies in the representation of the information involved. While binary relations over a set of alternatives are a canonical representation of preferences, a natural representation of judgments are binary valuations over a set of propositions, where the logical interconnections between these propositions determine the set of admissible valuations. E.g. the agenda of the famous discursive dilemma $\{p, q, p \wedge q\}$ is associated the set of admissible, i.e. logically consistent valuations $\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 1)\}$, where a 1(0) denotes a proposition to be believed (not believed).

The paper is structured as follows: In section 2 we introduce the formal framework. Section 3 discusses paradoxes of majority voting. We will use Saari's representation cubes to provide a unified geometric representation of profiles and majority rule outcomes and introduce Saari's idea of a profile decomposition. In this framework we will provide a characterization of profiles leading to the Anscombe paradox. Section 4 applies the same tools to judgment aggregation. In particular we show what drives the logical inconsistency of majority outcomes and how this can be avoided with the help of restrictions on the distribution of individual valuations, i.e. give a kind of generalized domain restriction. This leads us to the determination of the likelihood of inadmissible outcomes under majority rule for different agendas in section 5. In section 6, we apply our approach to illuminate current results on distance-based judgment aggregation. Finally, section 7 concludes the paper.

2. Abstract aggregation theory and majority voting

In the binary framework of abstract aggregation theory individual vectors of yes/no or true/false valuations $v = (v^1, v^2, \dots, v^{|J|}) \in \{0, 1\}^{|J|}$ from a set $X \subseteq \{0, 1\}^{|J|}$ of admissible valuations over a set J of issues (the agenda) are aggregated into a collective valuation. (In a slight abuse of notation we will use the term valuation both for a the binary valuation of a single issue as for vectors of binary valuations.)

Such an issue $j \in J$ might be the pairwise comparison between two alternatives in preference aggregation or a proposition on which a judgment needs to be made. Typically, the interconnections between the issues limit the set of admissible valuations. In judgment aggregation a valuation $v = (v^1, v^2, \dots, v^{|J|}) \in X \subseteq \{0, 1\}^{|J|}$, represents an individual's beliefs, where $v^j = 1$ means that proposition j is believed and X denotes the set of all admissible (logically consistent) valuations (see Dokow and Holzman, 2009).

Given a set N of individuals, a profile of individual valuations is then a mapping $p : N \rightarrow \{0, 1\}^{|J|}$ which assigns to each individual a vector of binary valuations. A desirable property of an aggregation rule, stronger than non-dictatorship, is of course anonymity, which requires that the same collective valuation be assigned to any permutation of the set of individuals.

If anonymity is assumed, a profile of individual valuations can be represented by a vector $\mathbf{p} = (p_1, \dots, p_{|X|}) \in [0, 1]^{|X|}$ with $\sum_k p_k = 1$, which associates with every admissible valuation $v_k \in X$ the share p_k of individuals with this valuation. Such an anonymous representation of profiles is particularly appropriate for the analysis of majority voting, where anonymity is typically assumed.

Geometrically any binary valuation is a vertex of the $|J|$ -dimensional hypercube and, more interestingly, any anonymous profile $\mathbf{p} \in [0, 1]^{|X|}$ can be given a lower-dimensional representation by a point $x(\mathbf{p}) \in [0, 1]^{|J|}$ in the $|J|$ -dimensional 0/1-polytope, i.e. the convex hull of the hypercube $\{0, 1\}^{|J|}$, where for each component $j \in J$, $x^j(\mathbf{p}) = \sum_{k \in \{1, \dots, |X|\}} p_k v_k^j$ denotes the average support for issue j . Thus the $|J|$ -dimensional 0/1-polytope will be referred to as the representation polytope of the profiles.

An abstract anonymous aggregation rule is a mapping f that associates with every anonymous profile $\mathbf{p} = (p_1, p_2, \dots, p_{|X|}) \in [0, 1]^{|X|}$ a valuation $v = f(x(\mathbf{p})) \in \{0, 1\}^{|J|}$.

We will write $v(\mathbf{p})$ for $f(x(\mathbf{p}))$ and identify by $v^j(\mathbf{p})$ the j th component of $v(\mathbf{p})$ under the given aggregation rule.

In this framework majority voting on issues (or majority voting for short) is defined as follows:

Definition 1 For any issue $j \in J$ and any profile $\mathbf{p} \in [0, 1]^{|X|}$, $M_{v^j}(\mathbf{p}) \in \{0, 1\}$ is the outcome of majority voting on issue j if $v^j(\mathbf{p}) = 1 \Leftrightarrow x^j(\mathbf{p}) > 0.5$.

This representation immediately provides majority with a wellknown metric rationalization in terms of the Hamming distance between binary vectors. (For any two binary vectors $v, v' \in \{0, 1\}^{|J|}$, the Hamming distance $d_H(v, v')$ is the number of components in which these two vectors differ.)

Proposition 1 (Brams et al., 2004) For any profile $\mathbf{p} \in [0, 1]^{|X|}$, the valuation ${}^M v(\mathbf{p}) \in \{0, 1\}^{|J|}$ is the majority outcome if and only if it minimizes the sum of Hamming distances weighted by the population shares, or formally,

$${}^M v(\mathbf{p}) = \arg \min_{v \in \{0, 1\}^{|J|}} \sum_{k=1}^{|X|} p_k d_H(v_k, v).$$

Thus, whenever the sum of Hamming distances can be interpreted as an appropriate measure of social disutility, majority voting can be justified by its minimization.

Observe however that nothing in this characterisation prevents the majority outcome ${}^M v(\mathbf{p}) \in \{0, 1\}^{|J|}$ from being an inadmissible valuation, i.e. that ${}^M v(\mathbf{p}) \in \{0, 1\}^{|J|} \setminus X$.

In the hypercube, a more natural metric representation of majority voting can be given in terms of the euclidean distance d_E .

Proposition 2 For any profile $\mathbf{p} \in [0, 1]^{|X|}$, the valuation ${}^M v(\mathbf{p}) \in \{0, 1\}^{|J|}$ is the majority outcome if and only if it minimizes the euclidean distance between the corresponding vertex and the point $x(\mathbf{p})$ in the representation polytope, or formally

$${}^M v(\mathbf{p}) = \arg \min_{v \in \{0, 1\}^{|J|}} d_E(x(\mathbf{p}), v).$$

Conversely, the set of all profiles for a given majority outcome $v \in \{0, 1\}^{|J|}$ defines a subcube of $[0, 1]^{|J|}$, $P_v = [v^j - 0.5]^{|J|}$, which is the set of all profiles for which v is the majority outcome. Such a subcube will be called the majority subcube of v (or simply v -subcube) and can be seen in Figure 1 for vertex $(1, 0, 1)$.

3. The Anscombe paradox and the irrationality of a metric rationalization

Because majority voting on issues has a metric rationalization in terms of distance minimization, it is quite disturbing that the majority outcome need not be the one that minimizes the distance for the majority of individuals,

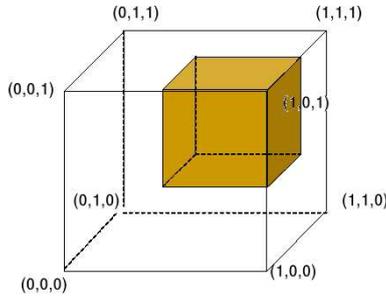


Fig. 1 — Majority subcube

as the Anscombe paradox shows. In other words the Anscombe paradox states that a majority of the voters can be on the losing side on a majority of issues. Formally, the Anscombe paradox can be defined in the following way:

Definition 2 A profile $\mathbf{p} = (p_1, \dots, p_{|X|}) \in [0, 1]^{|X|}$ exhibits the Anscombe paradox if

$$\sum_{k \in \{1, \dots, |X|\} : d_H(v_k, Mv(\mathbf{p})) > \frac{|X|}{2}} p_k > \frac{1}{2}.$$

Indeed, it is the particular distribution of individual valuations that leads to the paradox. To analyse this and further paradoxes in later sections, we will numerate the vertices of the three-dimensional hypercube as listed in Table 3.1.

valuation		valuation	
v_1	(0, 0, 0)	v_5	(1, 1, 0)
v_2	(1, 0, 0)	v_6	(1, 0, 1)
v_3	(0, 1, 0)	v_7	(0, 1, 1)
v_4	(0, 0, 1)	v_8	(1, 1, 1)

Table 3.1 — Valuations in three-dimensional hypercube

Now, consider the profile $\mathbf{p} = (\frac{2}{5}, 0, 0, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0)$, which specifies exactly an Anscombe paradox situation. It is easily observed, that $x(\mathbf{p}) = (\frac{2}{5}, \frac{2}{5}, \frac{2}{5})$ and hence the majority outcome is $Mv(\mathbf{p}) = (0, 0, 0)$.

Are we able to specify profiles that lead to an Anscombe type or other paradoxical majority outcome? Saari (2008) identifies what he calls ‘Condorcet portions’ as the driving part of paradoxes of preference aggregation.¹ In our three-dimensional setting for abstract aggregation problems, we can consider such portions as triples of valuations that have a common neighbor, i.e. a valuation that differs from each of the three valuations in exactly one issue.² Given that, we can now easily specify for every vertex in the hypercube its triple of neighbors. E.g. for v_5 the corresponding triple of neighbors is (v_2, v_3, v_8) . Table 3.2 indicates the triples for all eight vertices, the set of all such triples will be denoted by \mathcal{P} .

valuation	triple	valuation	triple
v_1	(v_2, v_3, v_4)	v_5	(v_2, v_3, v_8)
v_2	(v_1, v_5, v_6)	v_6	(v_2, v_4, v_8)
v_3	(v_1, v_5, v_7)	v_7	(v_3, v_4, v_8)
v_4	(v_1, v_6, v_7)	v_8	(v_5, v_6, v_7)

Table 3.2 — Triples of neighbors

To analyse the paradoxical outcomes and suggest restrictions to overcome them, we will use a profile decomposition technique developed by Saari (1995). From a majority point of view it is clear that two opposite valuations about an issue do cancel out, i.e. have no impact on the majority outcome. This can, however, be extended to any number of valuations by decomposing a profile into subprofiles:

Definition 3 For any profile $\mathbf{p} = (p_1, \dots, p_{|X|}) \in [0, 1]^{|X|}$ with $\sum_k p_k = 1$ a subprofile is a vector $\underline{\mathbf{p}} = (\underline{p}_1, \dots, \underline{p}_{|X|}) \in [0, 1]^{|X|}$ such that $\underline{p}_k \leq p_k$ for all $k \in \{1, \dots, |X|\}$.

It is obvious that the above decomposition argument for two opposite valuations does hold for any subprofile $\underline{\mathbf{p}}$ of \mathbf{p} for which $x(\underline{\mathbf{p}}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Such a subprofile does not influence the majority outcome based on \mathbf{p} at all.

As an example consider two individuals with the respective valuations v_2 and v_7 . They are exact opposites, so from a majority point of view those

¹A ‘Condorcet portion’ is a multiple of the set of individuals that has the following preferences over 3 alternatives a, b, c : $a \succ_1 b \succ_1 c$, $c \succ_2 a \succ_2 b$, $b \succ_3 c \succ_3 a$ leading to the majority cycle $a \succ b \succ c \succ a$.

²Equivalently we could say that they are each of Hamming distance 1 from their joint neighbor.

two valuations cancel out. Hence this implies that in any profile \mathbf{p} , for all opposite valuations we can cancel the share of the valuation held by the smaller number of individuals (and correct for the other shares accordingly) and still have the majority outcome unchanged.

Lemma 1 Let \mathbf{p} and \mathbf{p}' be two profiles such that, for $i \in \{1, \dots, 8\}$,

$$p'_i = \frac{\max\{p_i - p_{9-i}, 0\}}{\sum_{k=1}^4 |p_k - p_{9-k}|}.$$

Then $x^j(\mathbf{p}) \geq \frac{1}{2} \Leftrightarrow x^j(\mathbf{p}') \geq \frac{1}{2}$.

Proof The average support for each of the three issues can be stated as follows:

$$x^1(\mathbf{p}) = p_2 + p_5 + p_6 + p_8$$

$$x^2(\mathbf{p}) = p_3 + p_5 + p_7 + p_8$$

$$x^3(\mathbf{p}) = p_4 + p_6 + p_7 + p_8$$

Now, let $x^j(\mathbf{p}) = a$ and for some i , $|p_i - p_{9-i}| = t$, and assume w.l.o.g. that $1 > a \geq \frac{1}{2}$ and $0 < t < a$. For $\frac{a}{1} \geq \frac{1}{2}$ we also get $\frac{a-t}{1-2t} \geq \frac{1}{2}$. To see this suppose this is not the case, i.e. $\frac{a-t}{1-2t} < \frac{1}{2}$. It follows that $2a - 2t < 1 - 2t$. For $a \geq \frac{1}{2}$ this is false and therefore $\frac{a-t}{1-2t} \geq \frac{1}{2}$ is true. Repeat this for all $i \in \{1, \dots, 4\}$. For necessity just reverse the above arguments. \square

The lemma shows that in \mathbf{p}' at most 4 entries can be positive. As already previously mentioned, we can reduce a profile by any subprofile that does not change the majority outcome. Consider a subprofile $\underline{\mathbf{p}}$ with positive shares only for the valuations $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$, namely $\underline{\mathbf{p}} = (\frac{1}{8}, 0, 0, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, 0)$. On each issue there is the same number of individuals in favor of it and against it, i.e. $x(\underline{\mathbf{p}}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Hence, the elimination of such a subprofile does not change the majority outcome of the original profile and eventually increases the number of zero entries in the profile. The only two sets of valuations useable for such a reduction are $\{v_1, v_5, v_6, v_7\}$ and $\{v_2, v_3, v_4, v_8\}$.

Both, the pairwise reduction as well as the reduction using 4 valuations, lead to a reduced profile, the majority outcome of which is identical to the majority outcome of the original profile.

Now we can use the above concepts for a result in a three-dimensional framework, namely that the Anscombe paradox manifests itself in a particularly strong form:

Proposition 3 For $|J| = 3$ the Anscombe paradox will always show up in its *strong form*, i.e. a majority of the voters has a lower Hamming distance to the valuation which is the exact opposite of the majority outcome than to the majority outcome itself.

Proof Assume, w.l.o.g., that we want the majority outcome to be ${}^M v(\mathbf{p}) = (0, 0, 0)$. As $|J| = 3$, each voter k among a majority of the voters needs to have $d_H(v_k, {}^M v(\mathbf{p})) \geq 2$. Starting with ${}^M v(\mathbf{p}) = v_1$ this leads to the following conditions needed to be satisfied for the Anscombe paradox to occur, where the first three conditions guarantee the majority outcome to be v_1 and the fourth condition ensures that a majority of voters is of a Hamming distance of at least 2 from the majority outcome:

$$\begin{aligned} x^1(\mathbf{p}) &= p_2 + p_5 + p_6 + p_8 < \frac{1}{2} \\ x^2(\mathbf{p}) &= p_3 + p_5 + p_7 + p_8 < \frac{1}{2} \\ x^3(\mathbf{p}) &= p_4 + p_6 + p_7 + p_8 < \frac{1}{2} \\ p_5 + p_6 + p_7 + p_8 &> \frac{1}{2} \end{aligned}$$

Based on our previous decomposition argument (especially Lemma 1), no identical change in p_8 and p_1 would change the truth of any of the above inequalities. But this is also true for any other pair of opposite valuations. Hence we can directly look at the reduced profile \mathbf{p}' with at most 4 entries. For ${}^M v^j(\mathbf{p}') = 0$ it is not possible that more than half of the shares are located on one plane of the cube, i.e. $p'_8 + p'_r + p'_s < \frac{1}{2}$ for all $r, s \in \{5, 6, 7\}$. But this implies that $p'_r > 0$ for all $r \in \{5, 6, 7\}$ and hence $p'_1 > 0$ (and therefore $p'_8 = 0$ to enable ${}^M v(\mathbf{p}') = v_1$). Now for any $k \in \{5, 6, 7\}$, v_k is not closer to a majority of the voters' valuation than to ${}^M v(\mathbf{p})$, as $p_k < \frac{1}{2}$, and this would be the only voters with smaller distance. For any $k \in \{2, 3, 4\}$, v_k is not closer to a majority of the voters' valuation than to ${}^M v(\mathbf{p})$, as $p'_r + p'_s < \frac{1}{2}$ for all $r, s \in \{5, 6, 7\}$ and only two valuations out of $\{v_5, v_6, v_7\}$ are closer to v_k than to ${}^M v(\mathbf{p})$. \square

4. Judgment aggregation and the logical inconsistency of the majority outcome

In judgment aggregation, the issues in the agenda are logically interconnected propositions and thus not all valuations are admissible, i.e. logically consistent. Given the binary structure of the problem, we see that the tools of the geometric approach can be used to analyse paradoxes of judgment aggrega-

gation.³ The discursive dilemma with the agenda $\{p, q, p \wedge q\}$ and the associated set of admissible valuations $X = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 1)\}$ can again be analysed in our three-dimensional hypercube, in which the four admissible vertices determine the representation polytope as seen in Figure 2.

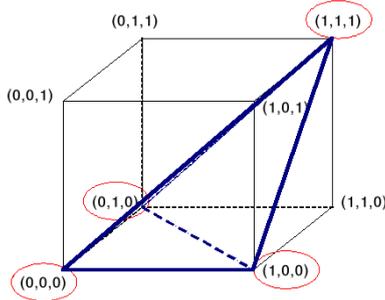


Fig. 2 — Representation polytope

Given the set of admissible valuations X , consider the profile $\mathbf{p} = (0, \frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0, \frac{1}{3})$, i.e. no voter has valuation $(0, 0, 0)$, one third of the voters has valuation $(1, 0, 0)$, and so on. As this maps into the point $x(\mathbf{p}) = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ - a point whose closest vertex is $(1, 1, 0)$ - the representation polytope obviously passes through the majority subcube of an inadmissible valuation, i.e. the set of admissible valuations X is not closed under majority voting. In this case X is called majority inconsistent.

That this type of paradox can easily occur with majority voting is seen from the following lemma:

Lemma 2 Given any vertex $v \in \{0, 1\}^{|J|}$, there exist 3 vertices v_a, v_b, v_c with respective shares p_a, p_b, p_c such that for some profile \mathbf{p} with $p_k = 0$ for all $k \notin \{a, b, c\}$, $x(\mathbf{p})$ lies in the v -subcube.

For $|J| = 3$, these 3 vertices necessarily need to have v as their common neighbor. Given that, we can now provide a simple result for the majority inconsistency of a set of valuations X , i.e. a necessary condition for X not to be closed under majority voting.

³See e.g. Saari (2008) for a very brief discussion of the link of his geometric approach to judgment aggregation.

Proposition 4 For $|J| = 3$, the set of admissible valuations X is majority inconsistent only if for some triple of vertices in the domain with a common neighbor, this common neighbor is not contained in the domain.

In our 3-dimensional setting, we can easily specify all possible triples that could lead to inadmissible majority outcomes. The reduced profile does have an interesting feature in exactly those situations when inadmissible majority outcomes could occur:

Proposition 5 For $|J| = 3$, if for some $v_i \in \{0, 1\}^3$ with $p_i \leq p_{9-i}$, each valuation in the triple of neighbors has a larger share in \mathbf{p} than its opposite valuation, then the reduced profile $\bar{\mathbf{p}}$ has at most 3 positive entries.

Proof Let $\mathbf{p} = (p_1, p_2, \dots, p_8)$ s.t. $\sum_k p_k = 1$. From Lemma 1 we know that

$$p'_i = \frac{\max\{p_i - p_{9-i}, 0\}}{\sum_{k=1}^4 |p_k - p_{9-k}|}.$$

Now in \mathbf{p}' there are at most 4 positive entries. Given that it is not possible that $[p'_i > 0 \wedge p'_{9-i} > 0]$ for any $i = 1, \dots, 8$, and that for some v_k each valuation in the triple of neighbors has a larger share than its opposite valuation, this only leaves two possibilities, namely that we have positive shares at most either for all of (p_1, p_5, p_6, p_7) or for all of (p_2, p_3, p_4, p_8) . However, in both cases - as was discussed before - further reductions are possible by looking for particular subprofiles. Let - for the above two combinations - $\mathcal{A} = \{i : p'_i > 0\}$ be all valuations for which there is a positive share. Then we can reduce the profile further to profile $\bar{\mathbf{p}}$ such that

$$\bar{p}_i = \frac{\max_{j \in \mathcal{A} / \{i\}} \{p'_i - p'_j, 0\}}{\sum_{i \in \mathcal{A}} p'_i - \min_{j \in \mathcal{A}} p'_j}.$$

Obviously $\bar{\mathbf{p}}$ has at most 3 positive entries. □

Example 1 Let us consider the following set of admissible valuations $X = \{v_1, v_2, v_3, v_8\}$, i.e. any profile $\mathbf{p} = (p_1, p_2, p_3, 0, 0, 0, 0, p_8)$, where $p_k \geq 0$ for all $k = 1, 2, 3, 8$ and $\sum_k p_k = 1$. As $v_1 = (0, 0, 0)$ and $v_8 = (1, 1, 1)$ are exact opposites, the reduced profile will have a share of 0 for the valuation held by the smaller number of individuals. In the case of $p_1 > p_4$ such a reduced profile will be $\bar{\mathbf{p}} = (\frac{p_1 - p_4}{p_1 + p_2 + p_3 - p_4}, \frac{p_2}{p_1 + p_2 + p_3 - p_4}, \frac{p_3}{p_1 + p_2 + p_3 - p_4}, 0, 0, 0, 0, 0)$, in the case of $p_1 \leq p_4$ we can create the reduced profile accordingly. Hence the reduced profile maps into one of the following two planes shown in Figure

3, namely either into the one determined by the vertices v_1, v_2 and v_3 or the one determined by the vertices v_2, v_3 and v_8 .

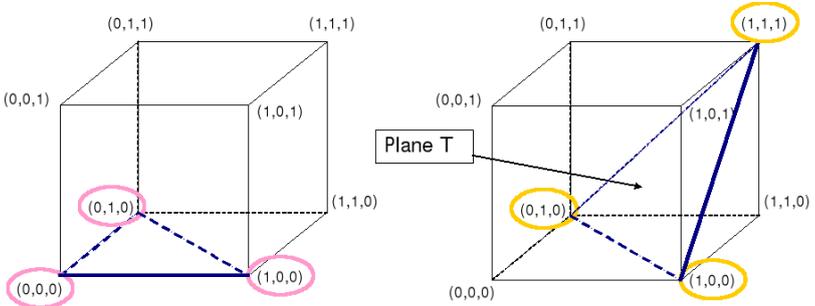


Fig. 3 — Planes

Problems may arise if the reduced profile has positive shares only for 3 valuations that constitute a triple in \mathcal{P} . For the above example this would be the triple (v_2, v_3, v_8) on the right side of Figure 3. Now, in Figure 4 the intersection of this plane with the $(1, 1, 0)$ majority subcube is indicated by the shaded triangle. Only if the reduced profile maps into this triangle do inadmissible majority outcomes arise.

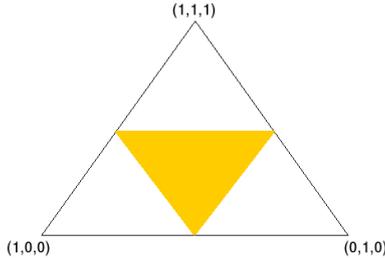


Fig. 4 — Plane T

Now, for the 3-dimensional framework we can state the following result:

Proposition 6 A set of admissible valuations $X \subseteq \{0, 1\}^3$ is majority inconsistent if and only if for some reduced profile $\bar{\mathbf{p}}$ the following conditions are met:

- $\bar{\mathbf{p}}$ has 3 positive entries

the 3 valuations with positive shares form a triple $(v_a, v_b, v_c) \in \mathcal{P}$ whose joint neighbor is not in X

the following condition holds for all $v_k \in \{v_a, v_b, v_c\}$ with corresponding shares $\bar{p}_k \in \{\bar{p}_a, \bar{p}_b, \bar{p}_c\}$:

$$\frac{\bar{p}_k}{\bar{p}_a + \bar{p}_b + \bar{p}_c} \leq \frac{1}{2}.$$

Proof The sufficiency part is obvious from Figure 4. For necessity, it is clear that with less than 3 positive entries in $\bar{\mathbf{p}}$ no inadmissible outcome can occur. Moreover, any triple not in \mathcal{P} is closed under majority rule. In the case of 4 positive entries in $\bar{\mathbf{p}}$ problems only arise in case a triple in \mathcal{P} has a positive share where the joint neighbor is not contained in X . But then the fourth positive entry must be one such that the resulting profile can still be further reduced and hence this contradicts the assumption that $\bar{\mathbf{p}}$ was the reduced profile already. Now, the only further option is a triple in \mathcal{P} with a joint neighbor not in X . In this situation inconsistency occurs exactly in the cut with the respective majority subcube (see Figure 4) whose points are specified by the condition above. \square

One interesting feature of this result is that the complementary set of profiles actually determines the domain that is closed under majority rule. As those restrictions are based on the space of profiles, this approach is more general than restrictions on the space of valuations which is usually used in the classical literature on domain restrictions. E.g. List (2005) introduces the unidimensional alignment domain which has a certain resemblance to Black’s single peakedness condition in social choice theory. It requires individuals to be ordered from left to right such that on each proposition there occurs only one switch from believing it to not believing it (or vice versa). For $|J| = 3$ a unidimensional alignment domain would not satisfy one of the above conditions for inadmissible majority outcomes.⁴

In addition, this geometric framework also opens a simple way to analyse various other paradoxical situations, e.g. strong support for one particular issue combined with an inadmissible majority outcome, as given in the following proposition:

Proposition 7 There exist profiles such that there is almost unanimous agreement on one issue and still an inadmissible majority outcome is obtained.

⁴For a more elaborated discussion on majority voting on restricted domains see also Dietrich and List (2007).

Proof Looking at Figure 4 one observes, that points close to the edge connecting the vertices $(1, 0, 0)$ and $(1, 1, 1)$ have almost unanimous agreement on issue 1. However, at the midpoint of this edge, the shaded triangle comes arbitrarily close to the edge. Hence, there exist profiles which lie in the shaded triangle but imply almost unanimous agreement on one issue. The same argument applies to points close to the edge connecting the vertices $(0, 1, 0)$ and $(1, 1, 1)$. \square

5. Likelihood of inadmissible majority outcomes

The geometric framework can also be used to analyze the likelihood of inadmissible majority outcomes in case $|X| \leq 4$. The approach is based on the fact that only 4 vertices are admissible individual valuations, and hence any point $x(\mathbf{p})$ in the representation polytope is determined by a unique profile \mathbf{p} . Consider again the situation $X = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 1)\}$. Then for any vector of shares of individual valuations $\mathbf{p} = (p_1, p_2, p_3, 0, 0, 0, 0, p_8)$ we get the following average support on each issue: $x^1(\mathbf{p}) = p_2 + p_8$, $x^2(\mathbf{p}) = p_3 + p_8$, $x^3(\mathbf{p}) = p_8$, $1 = p_1 + p_2 + p_3 + p_8$. As those are 4 equations with 4 unknowns there exists a unique solution. Thus, assuming every profile being equally likely - i.e. taking an impartial anonymous culture⁵ - the volume of certain subspaces now indicates the likelihood of occurrence of certain outcomes. Consider first the volume of the representation polytope V_R : $V_R = \frac{1}{2} \cdot 1 \cdot \frac{1}{3} = \frac{1}{6}$. On the other hand, points leading to inadmissible majority outcomes are located in the tetraeder determined by the points $[(\frac{1}{2}, \frac{1}{2}, 0), (1, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})]$. The volume of this tetraeder, V_T , is $V_T = \frac{1}{48}$ (see Figure 5).

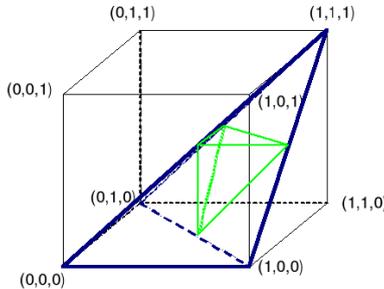


Fig. 5 — Distances

⁵ See Gehrlein (2006) for a general discussion of the impartial anonymous culture.

Now, the volume of the tetraeder relative to the volume of the whole representation polytope is $\frac{V_T}{V_R} = \frac{1}{8}$ and hence we can say that the probability of a majority outcome being inadmissible is 12,5 percent. This provides another - geometric - approach to derive the expected probability of paradoxical situations under impartial anonymous culture, leading to the same results as a previous approach by List(2005).

Of course, different domains allow for different probabilities. E.g. consider the agenda $\{p, q, p \leftrightarrow q\}$ with $X = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$. Then, for any point $x(\mathbf{p}) = (x^1, x^2, x^3)$ in the representation polytope we get $x^1(\mathbf{p}) = p_2 + p_8, x^2(\mathbf{p}) = p_3 + p_8, x^3(\mathbf{p}) = p_4 + p_8$ and $p_2 + p_3 + p_4 + p_8 = 1$. Again, every profile maps into a unique point in the representation polytope. Making the same volume calculations as before, we get - under the impartial anonymous culture - a probability of inadmissible majority outcomes of 25 percent.

6. Codomain Restrictions and Distance-Based Aggregation

We saw that using majority rule may lead to inadmissible majority outcomes. Restrictions on the space of profiles are one possibility to overcome such problems. An alternative way to guarantee admissible majority outcomes is to restrict the set of collective outcomes to admissible valuations.⁶ One way to work with such codomain restrictions is by using distance-based aggregation rules. Meskanen and Nurmi (2006) have provided an extensive analysis of distance-based aggregation rules in the Arrovian framework. In analogy to a well-known procedure in social choice theory (Kemeny, 1959), Pigozzi (2006) introduced such an approach to judgment aggregation. In principle a distance-based aggregation rule determines the collective valuation as the admissible valuation that minimizes the sum of Hamming distances to the individual valuations. Formally, this can be stated as follows:

$$f(\mathbf{p}) = \arg \min_{v \in X} \sum_{k=1}^{|X|} p_k d_H(v, v_k)$$

Given our geometric approach, there is a simple geometric explanation of this distance-based aggregation rule. As could be seen in Figure 4, all troublesome profiles lead to a point of average support in the shaded triangle. However, one option is to divide the triangle into three sub-triangles as in Figure 6.

⁶In social choice theory, aggregation rules based on such restrictions are often called Condorcet extensions.

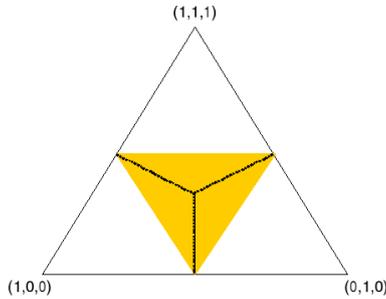


Fig. 6 — Distances

The intersection point of the three lines is exactly the barycenter point of the triangle, $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Those lines divide the shaded triangle into three equally sized sub-triangles, points in each sub-triangle are characterized by being of smallest Euclidean distance to the same vertex of the large triangle. E.g. points in the south-western sub-triangle will be closest to the $(1, 0, 0)$ vertex. As has been shown by Merlin and Saari (2000), the same will be obtained if - for any point in the shaded triangle - one switches the majority valuation on the issue j which is closest to the 50-50 threshold, i.e. for which $x^j(\mathbf{p})$ is closest to $\frac{1}{2}$. This will be illustrated in the following example:

Example 2 Let $X = \{(0,0,0), (1,0,0), (0,1,0), (1,1,1)\}$ and $\mathbf{p} = (0.1, 0.35, 0.3, 0, 0, 0, 0, 0.25)$. This leads to $x(\mathbf{p}) = (0.6, 0.55, 0.25)$ and hence an inadmissible majority outcome $M_V(\mathbf{p}) = (1, 1, 0)$. Looking at Figure 4 we see that $x(\mathbf{p})$ lies in the south-western sub-triangle. Thus, according to our distance-based aggregation rule, the outcome will be the admissible valuation $(1, 0, 0)$ as $x(\mathbf{p})$ is closest to the $(1, 0, 0)$ vertex. However, this can also be seen as switching the valuation on the proposition j whose average support $x^j(\mathbf{p})$ is closest to $\frac{1}{2}$. In $x(\mathbf{p})$ this is obviously proposition 2.

7. Conclusion

In this paper we have shown how geometry can be used to analyse paradoxes occurring under majoritarian aggregation and (impossibility) results in judgment aggregation, such as inadmissible majority outcomes and distance based aggregation rules. In addition we gave generalized domain conditions characterizing these paradoxes and determined the likelihood of such inadmissible majority outcomes.

Most of the stated results do not easily extend to more than three issues because of problems of dimensionality. E.g. an agenda with three propositions and their conjunction, like $\{p, q, r, p \wedge q \wedge r\}$, leads to eight admissible valuations, i.e. eight vertices out of the 16 vertices in the four-dimensional hypercube. The extensions of our (domain) restrictions and calculations of the likelihood of the occurrence of paradoxes to those higher dimensions are not obvious and need further work.

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