



A Preliminary Simulative Assessment of Disproportionality Indices

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Abstract What do indices of disproportionality actually measure? They provide an aggregate estimation of the difference between votes cast and seats assignment, but the relation between the value of the indices and the will of the voters is highly questionable. The reason is that when casting the vote the voter is deeply affected by the electoral system itself, possibly more deeply than s/he understands. The aim of this paper is to assess the performance of the most used indices of disproportionality with respect to the will of voters. To do so we compare by simulation their performance in some major electoral systems and with reference to some stylised typical cases. We use as a benchmark a ‘true’ index, i.e. an index that measures the difference between the will of the voters (instead of the votes) and the assignment of seats. In our experiment all the indices considered perform poorly, with the unexpected exception of the Loosemore-Hanby index.

Keywords electoral systems, disproportionality, agent based simulation

1. Introduction

In his fundamental work of 1999, Lijphart discusses at large the disproportionality of electoral systems.¹ His table 7.2 ranks 36 democracies according to the disproportionality of their system, as measured by the index of Gallagher, G. Not surprisingly, the most proportional system is the Netherlands, where the system is proportionality with a nation-wide constituency. Possibly not so expected is that the least one is France, where runoff should improve the proportionality with respect to plurality countries. Actually, the votes considered in France are those of the runoff ballot: this makes France a First-past-the-post country as for the computing of G, but for many voters the votes are second-best ones. This introduces our topic.

What do the index G and the other indices of disproportionality actually measure? The obvious answer is that they provide an aggregate estimation of the difference between votes cast and seats assignment. In turn, this difference is supposed to be an estimate of the difference between what could be defined the ‘aggregate will’ of the voters and the allocation of seats in the Parliament. It is *this* estimate that makes the index of interest. The distribution of votes is of interest because it is a proxy of the distribution of the preferences. If an index of disproportionality has a high value, the will of the voters is (supposed to be) poorly represented in the Parliament, while if its value is low the correspondence is (supposed to be) substantive.

Actually, the relation between the value of the index and the will of the voters is highly questionable. The reason is that when casting the vote the voter is deeply affected by the electoral system itself, possibly more deeply than s/he understands, as we will see. First, many voters will not vote for their preferred party if it is unlikely that it will gain a seat; they will prefer to turn to their second (or further) preference, or to abstain. The votes cast by second- (third-, etc.) best voters and by first- best voters are computed in the same way in the index, but the computing conceals a difference in the *representation of the will* that may be very large. The problem is even more serious if a voter abstains, as her/his will is simply *not represented* in the index, which by necessity considers only valid votes. This may produce a perverse result, because the abstention is likely to be higher if voters do not find a suitable party: a high rate of abstention indicates that the will of the voters is poorly represented, but their exclusion may result in a relatively low level of the index. Second, the very supply of parties is affected by the electoral system. As famously expressed by the Law of Duverger, non-proportional systems are likely to have *less parties* than proportional

¹ ‘Disproportionality occurs when political parties receive shares of legislative seats that are not equal to their shares of votes’ (Monroe, 1994, p. 138).

ones. Hence the choice of a voter in a plurality system is likely to be more constrained even if the voter is ready to vote for a party which is unlikely to win. In other terms, the *demand* of parties (or candidates) may be different, even very different, from its *supply*, and a voter may well be unaware of this difference. Her/his will may possibly be represented, but the will is forced to be a choice among a set of alternatives limited by the electoral rules. Possibly s/he does not even know that s/he could want more.

The distortion induced by the strategic voting and by the strategic supply of parties in non-proportional systems may be very high. Authoritatively, Cox (1997, p.97) claims that (in plurality systems) 'if clear information about candidate chances is provided, one can expect substantial levels of strategic voting, and a consequent reduction in the number of viable candidacies'. According to Alvarez et al. (2006), some 64% of the voters who could sensibly vote strategically actually did so in the British election of 1997. Also, data in the appendix show that in Western Europe more than 7% of the voters would have probably voted for a party located 1 or 10 on a ten-point left-right scale, and more than 16% for a party located lower than 3 or higher than 8. These parties are likely to be non present, or non credible, in a plurality system.

Both factors reduce the validity of the indices, and in an erratic way. We cannot know whether a voter voted for the Partial Freedom Party because s/he likes it or because it despises it, but less than the Pure Freedom Party. Nor we can know whether the same voter would have preferred the No Freedom Party, had it been present in the poll. Not by chance, all indices perform particularly well in Soviet-type elections, where there is only one party and voting is close to compulsory. This is not a joke: the constraints imposed by Soviet electoral law to the choice of voters may be considered the limiting case of a range that spans from the nearly-no-constraint case of one-district, no-threshold pure proportional system to plurality and beyond.² In addition, these constraints are different across countries and systems. The usual indices of disproportionality may fail their major aim, that is to be a tool suitable to measure the difference between the will of the voters and the composition of the Parliament; but possibly they are even less reliable if they are used to compare across real cases the performance of electoral systems with respect to the representation of the will of the voters.

How unreliable are the indices? The aim of this paper is to assess the performance of the most used ones. To do so we will compare their perfor-

²To be true, the cost of taking part in a poll induces some constraints even in fully proportional systems, and these constraints may be different in different countries. However these constraints are arguably minor with respect to the ones imposed by non-proportionality, and we will not consider them.

mance in some major electoral systems and with reference to some typical cases with that of a ‘true’ index, i.e. an index that measures the difference between the *will of the voters* (instead of the votes) and the assignment of seats. Details on the methodology are in the next section; the results are discussed in the following one. Section four contains our conclusions.³

2. Methodology

In the literature it is possible to find at least 25 indices of disproportionality (see among others Monroe, 1994; Taagepera and Grofman, 2003; Grilli di Cortona et al., 1999; Karpov 2008; Borooah, 2002). All of them are based on the difference between seats and votes, hence none of them is immune from the pitfalls described in the previous section. Possibly due to the unavailability of estimation tools (see below), the comparison of the indices has been typically performed according to their adherence to some consistency and implementability criteria defined *a priori* (see Pennisi, 1998; Taagepera and Grofman, 2003; Monroe, 1994; Karpov, 2008). As we wrote above, in this paper we will instead examine the performance of the indices considered through the comparison with a ‘true index’. We will deal only with some most well-known and most employed indices; further papers may provide a more complete overview. The indices considered in this paper are:⁴

$$\text{Gallagher (G), } [(1/2) \sum (v_i - s_i)^2]^{0.5}$$

$$\text{Lijphart (L), } (1/2) \sum |v_i - s_i|$$

$$\text{Loosemore and Hanby (LH), } (1/2) \sum |v_i - s_i|$$

$$\text{Rae (R), } (1/n) \sum |v_i - s_i|.$$

Where i refers to the parties, n is the number of parties, v is the share of votes and s the share of seats. The difference between LH and L is that L includes only the two majors parties. There has been some debate on how to treat minor parties in G (see Lijphart, 1994), but this problem is of no relevance for this paper.

To get rid of the constraints imposed by the electoral system (see above), we will consider different systems applied to the same set of preferences of the voters. In other terms, we will determine the assignment of seats

³ We must emphasize that the correspondence to the will of voters will be defined on a purely empirical basis. For a different approach (the satisfaction of given theoretical requirements) see Nurmi, 2005, and the literature quoted therein.

⁴ See Gallagher, 1991; Loosemore and Hanby, 1971; Lijphart, 1994; Rae, 1967.

produced by different electoral systems in a given case, and we will compare this assignment not with the votes cast, but with the first preferences of the voters. To do so we will consider the votes cast in a pure-proportional ballot with a nation-wide district as a proxy of the true preference of the voters. The ensuing assignment of seats provides the basis for the computing of the ‘true’ index of disproportionality. The index is simply

$$d_i = S'_i/S$$

where S is the total number of seats (the same for all electoral systems considered) and S'_i is the number of seats allocated by electoral system i differently from the allocation in the pure proportional, one-district system. For each electoral system considered the index is computed as

$$(1/2) \sum |S'_j - S_j|/S$$

where S'_j is the number of seats obtained by party j in that system, S_j is the number of seats obtained by party j in the pure proportional, one district ballot and S is the total of seats.⁵

To apply different electoral systems to the same set of preferences we resorted to a powerful program of electoral simulation, ALEX4, developed by M.E.Bissey at the University of Piemonte Orientale.⁶ The program requires as an input the number of seats, the share of first preferences of the voters for each party, the number of proportional districts and the number of voters per district; plus some parameters necessary to establish the full ordering of preferences, the geographical concentration of parties, and the propensity to strategic voting. Details on the inputs are in the appendix. The outputs of ALEX4 are the Parliaments as determined by 19 different electoral systems (majoritarian, proportional and parallel), the votes cast to each party in each electoral system, some indices of disproportionality (including index d_i above)⁷ and some indices of governability and of power (not relevant for this paper). A complete description of the program is in Bissey and Ortona, 2007. We considered three highly hypothetical cases and a less-hypothetical one, labelled *Virtual Italy*, *Virtual Netherlands*, *Virtual Europe* and *Real Italy*; and five electoral systems, i.e. one-district pure proportionality, threshold proportionality, Condorcet, Runoff majority, and Plurality, the last one both with and without strategic voting.⁸ The three

⁵ Note that rounding may produce a small deviation from 0 even in a pure-proportional, one-district system.

⁶ We used version 4.1.4.

⁷ Actually the indices provided are indices of *proportionality*; the output provided is $1 - d_i$.

⁸ See the appendix for details.

virtual cases have been obtained from the answers to question E033 of the European Value Survey of 2004, ‘In political matters people talk of “the left” and “the right”. How would you place your views on this scale, generally speaking?’. Each point of the ten-point left-right axis has been interpreted as a party.⁹ For the reasons explained in section 1, in non-proportional systems the convergence of parties towards the centres may have constrained the opinions of the responders, hence we assumed as cases for the study the two countries with a long-lasting tradition of proportional voting, Italy and Netherlands. To consider a different case we added the summary results of Western Europe;¹⁰ basic data are again in the appendix.

3. Results

Tables 1 to 4 present the percent results of the simulations for the four cases considered. In column 2 there is the ‘true’ index (see above), labelled T, and in the following ones the computed indices and (in black) the difference between T and the index of the column, both rounded to one decimal.

Table 3.1 — Results for the Virtual Netherlands

	T	G	L	LH	R
Condorcet	57	60.6 -3.6	32.9 24.2	56.9 0.1	11.4 45.6
Plurality	53	38.4 14.6	27.9 25.1	50.3 2.7	10.1 42.9
Plurality with strategic voting	56	43.1 12.9	30.1 25.9	55.9 0.1	11.2 44.8
Runoff majority	54	41.2 12.8	31.8 22.2	54.4 0.6	10.7 43.3
Pure proportionality	2	1.3 0.7	0.6 1.4	2.6 -0.6	0.5 1.5
Threshold proportionality	7	4.2 2.7	1.6 5.4	7.6 -0.6	1.5 5.5

⁹The *Real Italy* case is described in appendix.

¹⁰As results from the data in the appendix, the answers in the last case are actually more concentrated towards the centre, mostly on party 5. This supports the hypothesis of a bias in non-proportional countries.

Table 3.2 — Results for Virtual Italy

	T	G	L	LH	R
Condorcet	62	46.6 15.4	31.9 30.1	61.8 0.2	12.4 49.6
Plurality	64	48.0 16.0	33.9 30.1	63.9 0.1	12.8 51.2
Plurality with strategic voting	61	45.6 15.4	32.4 15.4	60.8 0.2	12.2 48.8
Runoff majority	68	50.9 17.1	38.4 29.6	67.8 0.2	13.6 54.4
Pure proportionality	1	0.9 0.1	0.4 0.6	1.5 -0.5	0.3 0.7
Threshold proportionality	8	5.3 2.7	2.4 5.6	9.0 -1.0	1.8 6.2

Table 3.3 — Results for Virtual Western Europe

	T	G	L	LH	R
Condorcet	67	50.3 16.7	89.7 -22.7	66.6 0.4	12.3 53.7
Plurality	69	73.4 -4.4	41.7 27.3	68.6 0.4	13.7 55.3
Plurality with strategic voting	68	72.2 4.2	40.7 27.2	67.6 0.4	13.5 54.5
Runoff majority	66	70.0 -4.0	39.2 26.8	65.6 0.4	13.1 52.9
Pure proportionality	2	1.4 0.6	0.8 1.2	2.2 -0.2	0.4 1.6
Threshold proportionality	14	6.7 7.3	3.8 10.2	13.4 0.6	2.7 11.3

Table 3.4 — Results for ‘Real’ Italy

	T	G	L	LH	R
Condorcet	37	26.4 10.6	18.4 18.6	36.8 0.2	8.2 28.8
Plurality	49	37.2 11.8	30.3 18.7	48.9 0.1	10.9 38.1
Plurality with strategic voting	34	19.8 14.2	17.1 16.9	34.2 0.2	7.6 26.4
Runoff majority	36	22.4 13.6	17.8 18.2	35.7 0.3	7.9 28.1
Pure proportionality	0.5	0.3 0.2	0.2 0.3	0.5 0.0	0.1 0.4
Threshold proportionality	13	7.0 6.0	4.6 8.4	13.5 0.5	3.0 10.0

These results suggest two relevant considerations (a and b below) and some less relevant ones.

a) The index LH performs remarkably well. Why LH performs better than the other indices is apparently easy to explain. As may be expected, and as results from our data, all the indices tend to underevaluate disproportionality (there are only seven minus signs in ninety-six figures, five of which in proportional systems). LH compensates for such underevaluation better than the other indices. It adds more terms than L (in our experiment, ten or nine instead of two), and it divides the resulting sum by a figure lower than in R (in our case, two instead of ten or nine). Finally, the figure for LH is generally higher, often largely higher, than that of G, which can again explain its better performance as a result of a better compensation for the underevaluation implicit in the index.

However, two points are intriguing. First, there are some cases in tables 1 and 3 where G is higher than LH, and in these cases too LH performs better. Second, and more relevant, LH not only performs better than the other indices; it also performs very well. Its largest difference from the 'true' index is under plurality in table 1, with an absolute difference of only 2.7 and a relative difference as small as 5%. Clearly, further inquiry is requested, both experimental/simulative and theoretical.

b) The other indices perform poorly in most cases. The second best performing index, G, obtains relatively good figures (an absolute difference lower than 10) only in proportional systems, as obvious, and in four other cases out of 16. Three are in virtual Western Europe; the most relevant difference of this setting with respect to the others is the presence of a large centrist party, which attracts both votes and seats. This suggests that G may provide good results in cases where few parties get votes; possibly deceitfully, if the choices of the voters are constrained -see the discussion in section 1. The performance of R is particularly poor; its underevaluation of the disproportionality is clearly due (in our case) to the high value of the denominator. This suggests that it can provide better results if there are less parties; again, these results may easily be deceitful due to what in section 1.¹¹

c) The hierarchy of the indices is clear, $LH > G > L > R$; this order is respected in all cases, bar an equal value of G and L and a better results of

¹¹ The correlation among all the indices is very high, due to the presence of two clusters of systems (proportional and non-proportional). If we consider only non-proportional systems, T correlates highly with LH ($r = 0.9943$) and R ($r = 0.9892$); less with G ($r = 0.875$) and L ($r = 0.6506$). All figures are significant at 0.01.

G, both in table 3 (the last one, however, in a proportional system, where all the indices perform reasonably well, as expected). This goes against a general feeling in the literature that G is the most reliable index.

d) The clusters of the results are generally respected, albeit the values are scaled down. However, some results are erratic: this is the case of G for Condorcet in tables 1 and 3, and of L again for Condorcet in table 3.

e) The correspondence between the order of the indices and that of T is quite sound; exceptions are L in tables 1 to 3, and G and R in table 3. Note however that the values of T in table 3 are very close. This suggests that the traditional indices, bar L, may possibly be used to compare different electoral systems across a given set of preferences, albeit with great caution. Further experiments should help.

4. Conclusions

Our experiment suggests that the three indices G, L and R perform very poorly as indicators of the misallocation of the seats with respect to the preferences of the voters. They should be employed with great caution, and no inference about the correspondence of seats and preferences should be drawn from that of seats and votes. LH makes a noticeable (and unexpected) exception; however, the reasons of its good performance in our experiment are unclear. Further analysis is strongly recommended. The opinion that G is the better index is not confirmed with reference to the distribution of the preferences of the voters. Finally, the indices, bar L, appear to be not too unreliable as a tool to compare *ordinally* electoral systems across a given case, arguably provided that the difference in the allocation of seats among the systems is not too small.

However, we must emphasize that our experiment is very preliminary. Further research should include a more complete plan of experiments, to consider a more general panoply of cases relevant both from the theoretical and the empirical point of view; and a comparative static analysis of the indices with respect to some basic features, like the propensity to strategic voting, the number of parties, the district magnitude, and so on. Advanced simulation programs like ALEX4 should allow to tackle both tasks.

Appendix: Input Data

a) ‘Virtual’ cases. Percent shares of first preferences.

Party (1, most leftist)	‘Virtual’ Italy	‘Virtual’ Netherlands	‘Virtual’ W. Europe
1	5.22	1.87	3.83
2	5.03	4.47	3.73
3	10.06	13.91	10.31
4	11.52	16.30	11.28
5	23.17	23.05	30.42
6	17.06	16.72	15.92
7	10.69	16.51	10.72
8	8.47	5.92	7.93
9	4.33	0.83	2.62
10	4.46	0.42	3.24

b) Details about the ‘Real Italy’ case.

Party (1, most leftist)	Share % of first preferences
1	8.4
2	2.2
3	33
4	2.7
5	3.8
6	7.8
7	24.5
8	12.8
9	4.8

We employed the data used in Ottone et al. (2007), that refer to the election of the *Camera dei Deputati* in 2006. There are 9 parties, with the distribution of first preferences as given in the table.

c) *Details about the simulations.* We supposed 100 (630, as this is the number of seats in the Italian *Camera dei Deputati*, in the ‘Real Italy’) uninominal districts, each with 100 voters. The resulting 10,000 (63,000) voters were collected in one overall district for proportional systems. The threshold of the threshold proportionality is 5%. Two parties take part in the second ballot of runoff majority. For plurality with strategic voting we assumed that in each district every voter votes for the biggest party of the coalition her/his preferred party belongs to. In the virtual cases, we assumed the presence of four coalitions (1 party – 4 parties – 4 parties – 1 party from the left to the right). In the ‘real’ scenario we assumed

the presence of two coalitions (5 parties – 4 parties from the left to the right).¹² The complete order of preferences for Condorcet voting has been generated by ALEX4. The procedure is the following. Each virtual voter chooses as the second preferred party an adjacent party with probability p_1 , a second-to adjacent party with probability p_2 , and another party at random with probability $1 - p_1 - p_2$. The procedure is iterated until the full order of preferences is generated. P_1 was arbitrarily established at 0.5, and p_2 at 0.1.

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¹²For other details about this simulation see Ottone et al. (2007).

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