

Necessary and Sufficient Conditions to Make the Numbers Count

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Abstract The article transfers an argument of Pattanaik and Xu on ranking opportunity sets to tragic choices and the so called "numbers problem". We characterize conditions that make the numbers count. This in itself will not resolve any problem relevant to the ongoing debate but should shed some fresh light on it by forcing participants to state specifically which of the assumptions (axioms) should give way for what reasons.

Keywords Aggregation, ethics, numbers debate

1. Introduction

The discussion of whether or not the numbers should count in ethical rankings of states of affairs has been going on for quite a while. It was preceded by related disputes like those over the 'trolley problem' (Thomson 1976), the survival lottery (Harris 1975) or more generally speaking utilitarianism's lack of respect for the separateness of persons (an old concern voiced forcefully in particular by Rawls (1971), see also Kliemt (1998)). But Taurek's (1977) problem of how to allocate an insufficient supply of a drug when six individuals are doomed if they do not get access to a sufficient quantity of that drug is, arguably, the most instructive one (and not burdened with standard action

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and omission problems).

In Taurek's example David needs all of the drug to survive while five other persons could each survive on one fifth of the supply. If in the name of substantive equality we should give nothing or exactly one sixth to each of the individuals we would be letting them all die. Throwing a fair coin would give each an equal survival chance of one half – with an expected value of the number of survivors of three. Allocating the full supply to David would rescue him for sure. Allocating the full supply in quantities of one fifth would rescue the five with certainty while ringing the bell on David.

Subsequently we do not intend to argue for or against any specific answer to Taurek's allocation problem. We suggest to look at the problem in a more indirect way by shifting the focus towards axioms that characterize the numbers' criterion. When ranking sets of individuals who are rescued or destined to perish the numbers' criterion is implied by three seemingly innocuous axioms. Anybody who does not agree that counting the numbers is the ethically correct way to rank sets of individuals (rescued or doomed) must reject at least one of the three axioms. Therefore discussing the normative appeal of each of the axioms may help us on in finding out what exactly is at stake in this context.

2. Axiomatic Framework¹

2.1 Preliminary Definitions

We consider an arbitrary but specific finite set X of individuals (human persons) who all will die unless by some external intervention a subset of these individuals be saved. Relevant decision problems will only emerge if X contains at least two persons. Individuals are indicated by small letters e.g. x, y, z . They are members (elements) of the finite set X of all individuals under consideration in the context at hand. Subsets of X are denoted by $\{x\}$ or $\{x, y\}, \{x, y, z\}$... or A, B, C . The set of all subsets of X including the empty set \emptyset is denoted by $\Pi(X)$.

We assume that a decision maker allocating a scarce resource to sets of individuals implicitly forms a (binary) moral preference relation on $\Pi(X) \times \Pi(X)$ that is defined by a subset R of $\Pi(X) \times \Pi(X)$.² Given any two finite sets A and B of individuals, the decision maker has to be able to say whether or not she deems it morally at least as good to save A as to save B (completeness). If she weakly prefers to save A over saving B this amounts to $(A, B) \in R$. Instead

¹In this section we transfer the axioms presented by Pattanaik and Xu (1990) to the framework used in the discussion on 'Do the numbers count?'

²In fact this may not be much of an 'assumption' as long as we have not required anything about the comparison. It is practically implied by any ranking activity.

of $(A, B) \in R$ we often write in the conventional manner $A R B$.

We assume that R is an ordering.³

Also in the conventional manner we define strict preference for $A, B \in \Pi(X)$ by $A P B : (A R B \& \neg B R A)$ and indifference by $A I B : (A R B \& B R A)$. We note that $A R B$ actually could be read as ‘rescuing A and dooming $X \setminus A$ ’ is weakly preferred to ‘rescuing B and dooming $A \setminus B$ ’ since the situation is specified such that an individual is rescued if and only if the individual is included in the chosen set.

In the following for a finite set A , $\#A$ denotes the cardinality of A .

2.2 Axioms for Ranking Sets of Individuals

Axiom 2.1 Indifference between Singletons

For all individuals $x, y \in X, \{x\} I \{y\}$.

According to this axiom, for all $x, y \in X$ the decision maker is indifferent between any two situations where she can save exactly one person, either person x or person y . Axiom 1 seems to be an almost unavoidable implication of the universalization requirement in passing moral judgement that is accepted in almost all ethical theory.⁴

Axiom 2.2 Simple Set Expansion Monotonicity⁵

For all individuals $x \in X, \{x\} P \emptyset$.

This axiom requires that to save one person’s life is better than saving no life.⁶ It may be seen as expressing minimum beneficence.

Axiom 2.3 Independence

³That R forms an ordering is, of course, a substantial assumption. A binary relation R on $\Pi(X) \times \Pi(X)$ is an ordering, if and only if for all $A \in \Pi(X) : A R A$ holds (reflexivity of the ranking R) and for all $A, B \in \Pi(X) : A R B$ or $B R A$ holds (completeness of the ranking R) and for all $A, B, C \in \Pi(X) : A R B$ and $B R C \Rightarrow A R C$ holds (transitivity of the ranking R). The proof of the central theorem, however, requires only transitivity. The other properties of an ordering are then implied by the axioms.

⁴Even if we are ethical non-cognitivist we may still use this axiom as a constitutive characteristic of moral discourse. Generalization in ethics is, of course, classically discussed in Singer (1971), with respect to utilitarianism in Hoerster (1971/1977), while its relation to the very concept of morals is analyzed in Singer (1973).

⁵It can be easily seen that the first axiom along with the premise that there is one individual whom it is better to rescue than not and transitivity implies axiom 2.

⁶Pattanaik and Xu (1990) in ranking sets of objects (rather than sets of human individuals) assumed for all $x, y \in X, x \neq y, \{x, y\} P \{y\}$.

For all $A, B \in \Pi(X)$ and for all $x \in X \setminus (A \cup B)$,

$$A R B \Leftrightarrow A \cup \{x\} R B \cup \{x\}.$$

Assume that the decision maker has ranked two sets A and B of individuals, such that she (weakly) prefers to save the lives of the individuals in A to saving the lives of those in B . Now another person x that does not belong to A and not to B joins both groups, i.e. x is saved together with the individuals in A and x is also saved if the individuals in B are saved. Then the ranking between A and B is always identical to the ranking of $A \cup \{x\}$ and $B \cup \{x\}$. Weak preference between A and B remains weak preference between $A \cup \{x\}$ and $B \cup \{x\}$, strict preference remains strict preference and indifference remains indifference.

One can also equivalently read the axiom ‘backwards’: Assume that in two sets C and $D \in \Pi(X)$ such that $C R D$, there exists the same individual $x \in C \cap D$. Now assume that x cannot be saved anymore (e.g. x has died), then this should not change the ranking between the remaining sets $C \setminus \{x\}$ and $D \setminus \{x\}$.

Axiom 3 may be seen as more demanding than the two preceding axioms since certain types of holistic interdependence are ruled out by it. In particular, the view that moving from C to $C \setminus \{x\}$ may harm a group C more than removing x from D such that $D \setminus \{x\}$ emerges with $D \setminus \{x\} P C \setminus \{x\}$ seems plausible. However, in view of the uniqueness of personality that we ascribe to individuals in *general* it is at least doubtful whether such ‘holistic’ effects should count for much. Moreover, since holistic effects should be seen as exceptions rather than the rule, the axiom would still be acceptable for all contexts in which such exceptions do not apply.

In the next step of our argument we will show that the preceding axioms are sufficient and necessary to make the numbers count. We show that the three axioms characterise a ranking of sets of individuals in $\Pi(X)$ that is the same as the one generated by the simple method to count the numbers.

3. Axiomatic Characterization of Counting the Numbers

Definition 3.1 For all $A, B \in \Pi(X)$, $A R_{\#} B :\Leftrightarrow \#A \geq \#B$.

Pattanaik and Xu (1990) prove a characterization theorem for the cardinality based ranking of non-empty sets of objects. In their version of the theorem only transitivity of the ranking has to be required, while in their proof reflexivity and completeness are implied by the other assumptions (c.f. Barberà, Bossert and Pattanaik 2003). We transfer the theorem to sets of individuals and in that process very slightly generalize the theorem and the proof to

include empty sets as well.

Theorem 3.1 Let R be a transitive binary relation on $\Pi(X) \times \Pi(X)$. R fulfils Indifference between Singletons, Simple Set Expansion Monotonicity and Independence if and only if $R = R_{\#}$.

Proof (1) It is obvious that a ranking of sets based on counting and comparing the numbers of individuals in sets is transitive and fulfils the three axioms.

(2) Let R be any transitive binary relation on $\Pi(X) \times \Pi(X)$ (recall that the corresponding strict relation is denoted by P and that of indifference by I) satisfying the three axioms. We have to show that for all $A, B \in \Pi(X)$

$$(2.1) \quad \#A = \#B \Rightarrow A I B$$

and

$$(2.2) \quad \#A > \#B \Rightarrow A P B.$$

(2.1) is proven by induction over the cardinality n of sets.⁷

Let x be any person then $\{x\} I \{x\}$ by Indifference between Singletons, implying $\{x\} R \{x\}$ by definition of I . Applying Independence means $\{x\} R \{x\} \Rightarrow \emptyset R \emptyset$, implying $\emptyset I \emptyset$. Thus we have stated the starting point of the induction for $n = 0$ and $n = 1$.

To prove the induction step from n to $n + 1$, assume that (2.1) is true for all sets $A, B \in \Pi(X)$ such that $\#A = \#B = n$ with $0 \leq n < \#X$. Consider $A, B \in \Pi(X)$ such that $\#A = \#B = n + 1$ with $0 \leq n < \#X$. Let $C \subset A$ be a subset of A such that $\#C = n$. Then there exists some $x \in A$ such that $A \setminus C = \{x\}$. Obviously $x \in B$ or $x \notin B$.

Case (i) $x \in B$. In this case define $D = B \setminus \{x\}$. Since $\#C = \#D = n$ we can apply the induction assumption for n to conclude that $C I D$ holds good. Since x is not contained in $C \cap D$, independence implies $A = C \cup \{x\} I D \cup \{x\} = B$.

Case (ii) $x \notin B$. Since $x \in A$ and $x \notin B$ and $\#A = \#B$, $B \setminus A \neq \emptyset$ has to hold and there exists a person $y \in B \setminus A$. Define $E := B \setminus \{y\}$. $\#C = \#E = n$ and by induction assumption it follows that $C I E$ holds. Independence implies $A = C \cup \{x\} I E \cup \{x\}$. Since $\#E > 0$ there exists an element $z \in E$. $\#[(E \cup \{x\}) \setminus \{z\}] = n = \#(B \setminus \{z\})$. The induction assumption implies $(E \cup \{x\}) \setminus \{z\} I B \setminus \{z\}$. Independence leads to $E \cup \{x\} I B$. We have $A I E \cup \{x\}$ and $E \cup \{x\} I B$ and and by transitivity $A I B$.

⁷The start of the induction in the proof by Pattanaik and Xu is $n = 1$. If $\#A = \#B$ $A I B$ is implied by Indifference between Singletons. In our case we have to deal with $\#A = 0$, too.

In order to prove (2.2) we choose any two sets $A, B \in \Pi(X)$ such that $\#A > \#B$. We can choose some subset $F \subset A$ such that $\#F = \#B$. (2.1) implies $F I B$. We define $G = A \setminus F$ which is a nonempty set. Let g be an element in G , then Simple Set Expansion Monotonicity implies $\{g\} P \emptyset$. Let f be in F then Independence implies $\{f, g\} P \{f\}$. Repeated application of Independence by adding all elements of F stepwise on both sides leads to $\{g\} \cup F P F$. Adding now all elements of $G \setminus \{g\}$ stepwise on both sides and applying Independence leads to $A = G \cup F P F \cup G \setminus \{g\}$. Analogously we receive for any other element h in $G \setminus \{g\}$ by Independence: $F \cup G \setminus \{g\} P F \cup G \setminus \{g, h\}$ and so on. Transitivity implies $A P F$ and together with $F I B$ we receive $A P B$ which completes the proof. \square

As we have shown, the inclusion of the empty set into the framework is possible and the characterization theorem still holds. That in one case we are discussing rankings of sets of human individuals and in one case rankings of sets of objects does not matter in the abstract characterization. Therefore we have now one answer to the question of when the numbers should count: i.e. whenever transitivity of the ranking and axioms 1–3 are accepted. The axioms are also independent as we will show next.

4. Independence of Axioms

This section illustrates the independence and the workings of the three axioms in bringing about the result. If we choose a subset of the axioms by omitting one of them there are possibilities to rank sets of individuals other than by counting numbers.

First we look for a transitive binary relation R_{queen} on $\Pi(X) \times \Pi(X)$ that fulfils Simple Set Expansion Monotonicity and Independence, but not Indifference between Singletons. Let us assume that there is a certain person q (queen or ‘David’) in the society such that whenever q belongs to a set A this is preferred to any set without q . Sets that either both contain q or both do not contain it are ranked according to numbers of persons. Formally R_{queen} is defined in the following way:

- (i) For all $A, B \in \Pi(X)$, such that $q \in A$ and $q \in B : A R_{queen} B \Leftrightarrow \#A \geq \#B$.
- (ii) For all $A, B \in \Pi(X)$, such that $q \notin A$ and $q \notin B : A R_{queen} B \Leftrightarrow \#A \geq \#B$.
- (iii) For all $A, B \in \Pi(X)$, such that $q \in A$ and $q \notin B : A P_{queen} B$.

R_{queen} is obviously complete and transitive and thus an ordering. R_{queen} does not fulfil Indifference between Singletons, since for any $x \neq q, x \in X \setminus \{q\} P_{queen} \{x\}$ holds because of (iii). R_{queen} fulfils Simple Set Expansion

Monotonicity. For all $x \in X$, $x \neq q$, $\{x\} P_{queen} \emptyset$ holds because of (ii), and $\{q\} P_{queen} \emptyset$ is implied by (iii). R_{queen} fulfils Independence. Let $A R_{queen} B$ be given and $x \in X \setminus (A \cup B)$.

If $x = q$, A and B are ranked with respect to numbers (i). $A \cup \{q\}$ and $B \cup \{q\}$ are also ranked with respect to numbers (ii). This means that $A R_{queen} B \Leftrightarrow A \cup \{q\} R_{queen} B \cup \{q\}$ holds.

Let $x \neq q$ be given. In cases (i) and (ii) of the definition we get the comparison by numbers and hence $A R_{queen} B \Leftrightarrow A \cup \{x\} R_{queen} B \cup \{x\}$. In case (iii) $A P_{queen} B \Leftrightarrow A \cup \{x\} P_{queen} B \cup \{x\}$ holds since $q \in A$ and $q \notin B$.

From this example we can conclude that as soon as we give up the axiom of Indifference between Singletons privileging individuals becomes possible. Not only that single persons are treated differently, it is also possible to rank groups according to whether or not a certain person belongs to a group.⁸

Rejecting Indifference between Singletons goes against the grain of universalistic ethical theory. If we want to avoid this, what about the other axioms? The next example contains a ranking R_{ind} that fulfils Indifference between Singletons, Independence, but not Simple Set Expansion Monotonicity. Giving up Simple Set Expansion Monotonicity means that there may be cases where saving nobody is at least as good as saving some person. Such a view could be the result of evaluating the trade off between equality of treatment or fairness of outcomes for the individuals and efficiency of the decision in favour of equality or fairness (cf. Broome 2002).

For all $A, B \in \Pi(X)$ we define $A R_{ind} B$, which represents the complete indifference ordering on $\Pi(X)$. If indifference between all sets (including the empty set) applies it is obvious that Indifference between Singletons, Independence, but not Simple Set Expansion Monotonicity are fulfilled.

In addition, we define the inverse ordering $R_{\#inv}$ which compares all $A, B \in \Pi(X)$ by defining $A R_{\#inv} B \Leftrightarrow \#A \leq \#B$. As a binary relation this ordering is identical to $R_{\#}$, however, the interpretation is inverse.

Proposition 4.1

$R_{\#}$, $R_{\#inv}$ and R_{ind} are the only orderings that fulfil Indifference Between Singletons and Independence.

Proof

It is easy to show that the three rankings have the desired properties. The inverse direction is shown in two steps.

- (a) If Simple Set Expansion Monotonicity holds the only candidate is $R_{\#}$.

⁸One could also assign a certain value to each person and rank sets of individuals with respect to the sum of the values assigned to the persons in each set. For the case of ranking sets of opportunities this proposal was modelled by Ahlert (1993).

(b) Assume Simple Set Expansion Monotonicity does not hold.

Case (b₁) There is at least one x in X such that $\{x\} I \emptyset$. We prove the following statement by induction: For all finite sets of cardinality n such that $\#X \geq n > 0$ it holds that they are indifferent to the empty set.

$n = 1$: Indifference between Singletons implies $\{x\} I \{y\}$ for all $x, y \in X$ therefore by transitivity $\{y\} I \emptyset$ for all $y \in X$.

Assume the statement holds for $\#X > n \geq 1$ and let a set $A \subseteq X$ with $\#A = n + 1$ be given and $y \in A$, then by induction assumption $A \setminus A \setminus \{y\} I \emptyset$. Independence yields $A \setminus A \setminus \{y\} \cup \{y\} I \emptyset \cup \{y\}$ and this, since $\emptyset \cup \{y\} = \{y\}$ and $\{y\} I \emptyset$, by transitivity $A I \emptyset$. Since all sets are indifferent to the empty set transitivity implies that all sets are indifferent. Therefore in case (b₁) the only ranking is R_{ind} .

Case (b₂) For all individuals $x \in X$, $\emptyset P \{x\}$. This leads to the ordering which is equivalent as a binary relation to $R_{\#}$ but compares inversely. ‘Always save the smaller number of individuals.’ \square

The next counterexample is an ordering R_S , that fulfils Indifference between Singletons, Simple Set Expansion Monotonicity, but not Independence. Let us assume that the only important aspect is that lives are saved, but the number of lives does not count. Define

- (i) for all $A, B \in \Pi(X)$ such that $A \neq \emptyset$ and $B \neq \emptyset : A I_S B$.
- (ii) for all $A, B \in \Pi(X)$ such that $A \neq \emptyset$ and $B = \emptyset : A P_S B$.

R_S is an ordering that fulfils Indifference between Singletons because of (i) and Simple Set Expansion Monotonicity because of (ii). R_S does not fulfil Independence, since $\{x\} P_S \emptyset \Leftrightarrow \{x, y\} P_S \{y\}$ does not hold for some $y \neq x$ in X which exists, since $\#X > 1$.

5. Conclusion

We have not shown that the numbers should count. We have shown that those who think that the numbers should not count must reject one of three axioms. Rejecting axiom 1 (Indifference between Singletons) seems almost tantamount to giving up universalistic ethical theory. Though ethical particularism may well be the better alternative those who go that route to avoid letting the numbers count should be aware what they are doing. Rejecting axiom 2 (Simple Set Expansion Monotonicity), besides counting, merely allows for complete indifference between all alternatives or always preferring the smaller number. Therefore rejection of Simple Set Expansion Monotonicity

will not lead to much. Rejection of axiom 3 (Independence) as the remaining possibility should attract most attention. Within universalistic ethical approaches (sticking to axiom 1) the relative merits of forming moral judgments in ways conforming with or violating axiom 3 should be discussed if we want to know whether the numbers should count. Ethical universalists who think that the numbers should not count have to be either morally indifferent between all – including zero – numbers of rescued (or doomed) individuals or must reject Independence. The first alternative seems weird while the holistic connotations of the second do not cohere well with the unique value assigned to the individual person which gives rise to Taurek's numbers problem in the first place.

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