



## An Axiomatic Characterization of a Boundedly Rational Bargaining Solution Based on Two Egalitarian Principles

Marlies Ahlert

Faculty of Economics, Martin-Luther-University Halle-Wittenberg  
(e-mail: ahlert@wiwi.uni-halle.de)

Bodo Vogt

Faculty of Economics, Otto-von-Guericke-University Magdeburg  
(e-mail: bodo.vogt@WW.Uni-Magdeburg.de)

**Abstract** For an axiomatic approach to a special solution of the Nash-bargaining problem we use a weakened form of invariance of the solution under affine linear transformations of utility which is motivated by behavioural studies on bounded rationality. Besides standard axioms a new compensation axiom is applied. We uniquely characterize a solution which models a rule proposed in the Talmud. This solution is also supported by experimental studies.

*JEL Classification* C78, C90

*Keywords* axiomatic bargaining, Talmud rule, bounded rationality

### 1. Introduction

The Nash bargaining problem is one of the most often considered problems in economics. Since the first articles of Nash (1950 and 1953) several solutions to this problem have been proposed (for a review see Thomson 1994 and references therein). If we assume interpersonally non comparable utilities, the most prominent ones are the Nash solution (Nash 1950 and 1953), the Kalai–Smorodinski solution (Kalai and Smorodinsky 1975, Kalai 1977), and other proportional solution concepts. These solutions have been characterised by different types of axiomatic approaches. These approaches explicitly or implicitly use the axiom of invariance of the solution under positive

affine transformations of the utility functions of the players. This means that multiplying the utility of a player by a positive constant or adding a constant should not change the solution of the economic bargaining situation.

Some textbooks give the following motivation for this axiom. Whether the bargaining problem is played in \$ or in another currency should not change the economic result. This argument would imply invariance of the solution under multiplication with a positive constant. It is also argued that giving persons some additional amount of money should not change the result of the pure bargaining situation. This implies invariance under adding a constant to the payoff of each person. One problem arising from these motivations is that in the arguments monetary payoffs are considered, whereas utilities should be considered.

From the theoretical perspective of the Nash bargaining program the axiom of invariance under positive affine transformations of the player's utilities is necessary because it is assumed that the evaluation of the economic outcomes by each player is represented by a cardinal utility function. This evaluation can be represented by any positive affine transformation of a given utility representation equivalently. Since utilities are assumed to be interpersonally non comparable, different positive affine transformations can be applied to different players' utilities.

In this paper we take a different view on the representation of the economic bargaining outcomes. Let us assume that the bargaining problem is defined in monetary terms. Then we consider the case that the persons have von Neumann–Morgenstern like utility functions for money which are of the form  $u(x) = \log(x)$ . Utility functions of this type have been proposed by Bernoulli (1738) and have been used by several other authors (like for example Allais 1953) in economic theory. A “behavioral” motivation of this kind of utility function is given by the Weber–Fechner Laws (from 1834 and 1860, reprinted in Fechner 1969) of psychophysics. These state a logarithmic perception of psychophysical stimuli. In textbooks on Marketing for example, this is often considered as a motivation for logarithmic perception of prices. (Arrow–Pratt risk aversion is constant for logarithmic utility functions as also for functions  $x^\alpha$  proposed in prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992). These are the only two types of functions showing constant risk aversion.)

If we assume that the person's evaluations of monetary payoffs can be represented by logarithmic utility functions, then what will happen to our text book argument? Clearly adding monetary payoffs to a bargaining problem changes the result, because the logarithm is not a linear function. The argument that bargaining in \$ should be equivalent to bargaining in any other currency remains. Multiplication of payoffs by positive constants, however, is

transformed by the logarithmic utility function into adding constants to the utility (and not multiplying the utility with a constant). One of the two requirements remains: invariance of the solution if a constant is added to any utility function.

What type of operation with monetary payoffs would imply a multiplication of utilities by a constant? Using the rules for logarithms it follows that the monetary payoffs have to be taken to the power of the constant. Even if one assumes that decision makers are mentally capable of performing this operation it does not seem to be reasonable to assume that the economic solution should be invariant if payoffs are taken to any power.

Let us look at the example of 2 persons that have to agree on one of the two payoff pairs which are  $(0.5, 1.5)$  and  $(1, 1)$  (with the first component of the vector being the payoff of person 1 and the second component being the payoff of person 2) given a status quo of  $(0, 0)$ . It is known from many experimental studies on bargaining agreements that in this situation the rule of equal payoffs plays a dominant role in determining the outcome. But if one takes the payoffs to the power of 1000 the choice is between  $(0.000, 6^{25})$  and  $(1, 1)$ . In this case there is a lot of experimental evidence that  $(0.0, 6^{25})$  is selected. The choice changes by taking payoffs to the power of a constant. Therefore, multiplying utilities by constants might change the choice of the alternative the players agree to. Other experimental studies (Vogt and Albers 1997 and 2001, Vogt 1999) support that it is necessary to weaken the assumption of invariance in order to fit the model to the observed behavior of subjects.

In section 2 of this paper we will present a two person bargaining approach with a bliss point that uses only the weak form of invariance with respect to adding constants to the player's utility functions. In addition to this axiom we use some standard axioms of cooperative bargaining theory, symmetry, Pareto optimality and restricted monotonicity. To obtain a unique solution we define a new axiom that is called independence of compensation. The motivation for this axiom is derived from the comparison of the strength of two aspiration levels of a person, its payoff in the status quo and its payoff in the bliss point. In section 3 we define a solution concept that can be interpreted as a compromise between two egalitarian principles. We uniquely characterise this solution by the set of axioms mentioned above.

The solution we obtain is also described in the Talmud (cf. Hokari and Thomson 2000 for a different approach to characterise this rule). Therefore, we call it the Talmud rule. It is empirically supported by an experimental study (Vogt and Albers 2001) of a bargaining problem. In this study an ex-ante logarithmic utility transformation of payoffs (without fitting parameters) is integrated into the analysis of the data. In contrast to other point pre-

dictions like the Nash solution (Nash 1950 and 1953), the egalitarian solution (Kalai 1977), the equal-loss solution (Chun 1988), and the Kalai–Smorodinsky solution (Kalai and Smorodinsky 1975, Kalai 1977), the Talmud rule is a good predictor for the experimental data mentioned above. We will conclude with some comments on the normative and descriptive aspects of the solution in section 4.

## 2. Notation, definitions, and axioms

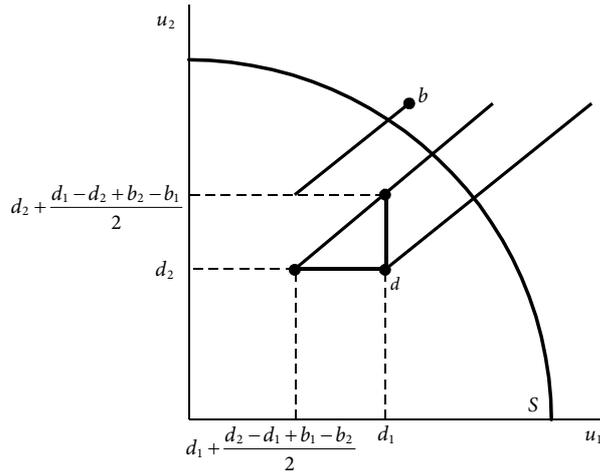
We consider bargaining situations with a status quo and a bliss point. The bliss point is in the literature on bargaining theory often called claims point. We interpret a player's payoff or utility in the status quo or in the bliss point as two aspiration levels that he uses to evaluate different bargaining proposals. (For an early definition of aspiration levels in boundedly rational bargaining theory cf. papers by Tietz and co-authors, 1978) The status quo aspiration level can be seen as a boundary to break off negotiations or as the payoff in case the negotiation fails. The bliss point can be derived by some claims that each player has developed from the embedding of the problem in some economic context. There might even exist some justifications for these aspirations. They can also reflect his goals in the negotiation or the payoff he or she sees as attainable. We assume that the bliss point aspiration level is individually feasible. Typically, however, the combination of the bliss point levels of both players is not feasible. From this fact partly the conflict of the negotiation arises.

*Definition* A triple  $(S, d, b)$  is called a 2-person bargaining situation, if

- (i)  $S \subset \mathfrak{R}^2$  and  $S$  is convex, compact and comprehensive.
- (ii)  $d = (d_1, d_2) \in S$  and there is  $s \in S$  such that  $s > d$ .
- (iii)  $b = (b_1, b_2) \in \mathfrak{R}^2$  such that  $b > d$  and  $b$  is individually feasible, i.e.
  - $\exists s = (s_1, s_2) \in S$  such that  $s_1 = b_1$ , and
  - $\exists t = (t_1, t_2) \in S$  such that  $t_2 = b_2$ $S$  is called the feasible set,  $d$  is the status quo and  $b$  is the bliss point.
- (iv) We assume that for given  $(S, d, b)$ :  $(d_1 + (d_2 - d_1 + b_1 - b_2)/2, d_2)$  and  $(d_1, d_2 + (d_1 - d_2 + b_2 - b_1)/2) \in S$ .

Conditions (i) and (ii) are standard for a model of a bargaining. Condition (iii) describes the individual feasibility of player's bliss point aspiration level. Condition (iv) means that an individually rational compromise between two egalitarian principles is feasible in  $S$ , equal gain compared to the status quo and equal loss compared to the bliss point (see Figure 1).

Figure 1



$B^2$  denotes the set of all 2-person bargaining situations fulfilling conditions (i) to (iv).

*Definition* A mapping  $F = B^2 \rightarrow \mathfrak{R}^2$  such that  $F(S, d, b) \in S$  for all  $(S, d, b) \in B^2$  is called a bargaining solution on  $B^2$ .  $F$  assigns a point in the feasible set to any given bargaining situation in  $B^2$ .

We want a solution  $F$  on  $B^2$  to fulfill the following axioms.

*Symmetry* Let  $(S, d, b) \in B^2$  be a 2-person bargaining situation such that  $S$  is a symmetric set with  $d_1 = d_2$  and  $b_1 = b_2$ . Then,

for  $F(S, d, b) = [F_1(S, d, b), F_2(S, d, b)]$ ,  $F_1(S, d, b) = F_2(S, d, b)$  holds.

If the representation of the bargaining problem is symmetric with respect to the information that is used in the space  $B^2$  (i.e. the feasible set, the status quo and the bliss point), then the solution point should also be symmetric.

*Pareto Optimality* For any  $(S, d, b) \in B^2$ ,  $F(S, d, b)$  is a weakly Pareto optimal point in  $S$ .

This Axiom is the standard efficiency requirement in bargaining theory.

*Restricted Monotonicity* Let  $(S, d, b)$  and  $(T, d, b)$  be 2-person bargaining situations in  $B^2$  such that  $S \subset T$ . Then  $F(S, d, b) \leq F(T, d, b)$  holds.

If the representation of a bargaining problem changes such that the feasible set is enlarged whereas the status quo and the bliss point remain unchanged, then the results for the bargaining parties do not decrease. A variant of this axiom was used by Roth (1979) to characterise the Kalai–Smorodinsky solution. In Klemisch-Ahlert (1996) experimental studies on bargaining behavior and a more general proportional solution concept of bargaining with goal functions is developed. The solution presented there is characterised using a similar type of monotonicity axiom which could be observed in player's bargaining behavior.

*Invariance Under Shifting* For all  $a = (a_1, a_2) \in \mathfrak{R}^2$  and all  $(S, d, b) \in B^2$  we define a situation  $(S + a, d + a, b + a)$  by

$$\begin{aligned} S + a &= \{(s_1 + a_1, s_2 + a_2) \mid (s_1, s_2) \in S\} \\ d + a &= (d_1 + a_1, d_2 + a_2) \\ b + a &= (b_1 + a_1, b_2 + a_2). \end{aligned}$$

Clearly  $(S + a, d + a, b + a)$  is again an element of  $B^2$ . Then,

$$F(S + a, d + a, b + a) = F(S, d, b) + a = (F_1(S, d, b) + a_1, F_2(S, d, b) + a_2)$$

If we shift a bargaining situation then the solution is transposed in the same way.

We restrict the invariance under positive affine transformations to invariance under shifting. This restriction is based on the results of behavioral studies on boundedly rational behavior, especially on models of the numerical perception. One argument for this restriction has been given in the introduction. If one assumes logarithmic utility functions multiplication of the utility with a positive constant corresponds to taking monetary payoffs to the power of this constant. This drastically changes the outcome of a bargaining problem as has been shown by the example in the introduction. This restriction is motivated by the fact that in monetary perception persons seem to react to differences of perceived monetary payoffs (utilities) and seem to be able to indicate when the utility difference between two payoff pairs is equal, but they do not seem to be able to deal with monetary payoffs that correspond to 10 times the utility of a given payoff. Deducing utility functions from lottery evaluations or decisions in games gives the result that multiplying utilities with positive constants changes the result of the evaluation, respectively of

the game. A model of the perception and many experimental studies supporting the model can be found in (Albers 1997, Vogt and Albers 1997 and 2001 and references therein). Therefore we only use this weaker form of invariance.

This means that the economic solution is independent of the representation of the bargaining problem in the space  $B^2$ . Equivalent representations can be interpreted as being generated by proportional payoff situations and their respective perception in utility space.

The following axiom is decisive for the type of solution concept we are going to characterise. It compares two situations where there has been a simultaneous change in one person's perception of the status quo and the bliss point.

We consider changes of these perceptions in opposite directions such that the differences are of the same absolute value. We demand that in this case the solution does not change. The idea is that if e.g. the perception of the status quo of person  $i=1,2$  increases by  $a_i > 0$  and the perception of the bliss point decreases by the same amount, then the strength of the bargaining position of person  $i$  remains unchanged.

With respect to one aspiration level the person's position becomes stronger, but this fact is compensated by a weakening of the other aspiration level. This also holds if some compensations are applied to both persons simultaneously.

*Independence of Compensation* Let  $(S, d, b)$  be a 2-person bargaining situation in  $B^2$  and  $a = (a_1, a_2) \in \mathfrak{R}^2$ . If  $(S, d + a, b - a) \in B^2$ , then  $F(S, d + a, b - a) = F(S, d, b)$ .

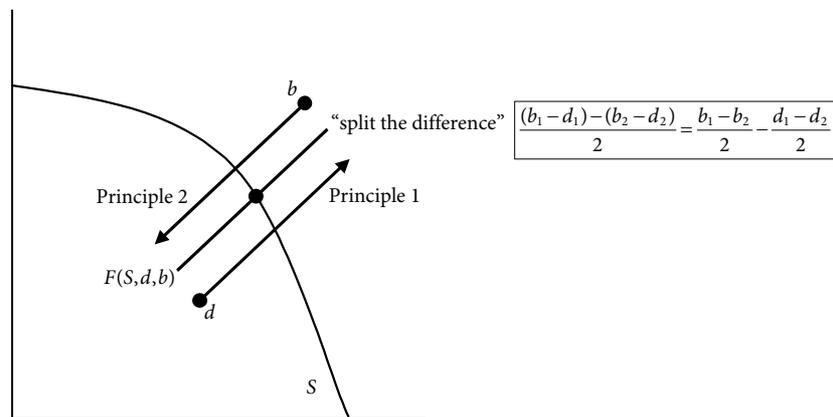
Special cases are individual compensations, i.e.  $a = (a_1, 0)$  or  $a = (0, a_2)$  with  $a_1, a_2 \in \mathfrak{R}$ .

### 3. The solution

We now define a solution concept  $F$  that incorporates two principles. The first one is the principle of equality of utility differences between the solution and the status quo for both persons. The other principle is the equality of the utility differences between the bliss point and the solution for both persons. Since it is not always possible to find a solution point such that both principles are fulfilled the concept applies a "split the difference" rule.

This rule then defines a line of pairs  $(x_1, x_2)$  which we call the "split-the-difference" line that is characterised by

Figure 2



$$x_1 - d_1 = x_2 - d_2 + \frac{b_1 - b_2}{2} - \frac{d_1 - d_2}{2} \quad \text{or} \quad x_1 = x_2 + \frac{d_1 - d_2 + b_1 - b_2}{2} .$$

These points can be seen as compromises between the egalitarian principle of equal gain compared to the status quo and equal loss compared to the bliss point.

The solution  $F$  is defined by the point of intersection of this line and the Pareto frontier of  $S$  and is therefore the maximal compromise in  $S$ . Under assumption (iv) in the definition of  $B^2$  we know that a individually rational compromise is feasible. Since  $S$  is compact a maximal compromise exists, i.e.  $F$  is well defined and  $F(S, d, b) \geq d$ .

It is straight forward to prove that  $F$  fulfills the axioms. Symmetry of the bargaining situation implies  $d_1 = d_2$  and  $b_1 = b_2$ . Therefore the “split the difference line” is the 45° line. This implies the symmetry of the solution. Pareto optimality follows directly from the definition of  $F$ . the two bargaining situations that are compared in the axiom of restricted monotonicity have the same “split the difference line”. Both solutions are points on this line and therefore the one of the enlarged set is in each component at least as large as the one of the included set. A shift of the whole bargaining situation shifts the “split the difference” line in the same way and therefore the solution, too.  $F$  is independent of compensation, since  $S$  is still the same feasible set and the “split the difference” line does not change under compensation:

$$x_1 = x_2 + \frac{d_1 + a_1 - (d_2 + a_2) + b_1 - a_1 - (b_2 - a_2)}{2}$$

$$\Leftrightarrow x_1 = x_2 + \frac{d_1 - d_2 + b_1 - b_2}{2}$$

The more difficult task is to show that  $F$  is the only solution on  $B^2$  that fulfils the set of axioms above. This means that we still have to prove the uniqueness part of the following theorem.

*Theorem*  $F$  is uniquely characterised by the axioms of Symmetry, Pareto Optimality, Restricted Monotonicity, Invariance under Shifting and Independence of Compensation.

*Proof* Let  $F$  be any solution fulfilling the set of axioms in the theorem and let  $(S, d, b)$  be a given situation in  $B^2$ .

First we apply the following shifting to the situation. We consider

$$(S + (0, c), d + (0, c), b + (0, c)) \text{ with } c = \frac{d_1 - d_2 + b_1 - b_2}{2}.$$

We denote this situation by  $(S', d', b')$ . It is an element of  $B^2$  and  $F(S', d', b') = F(S, d, b) + (0, c)$ .  $(S', d', b')$  has the property  $d_1' + b_1' = d_2' + b_2'$ , since  $d_2' + b_2' = d_2 + b_2 + 2 \cdot \frac{1}{2} \cdot (d_1 - d_2 + b_1 - b_2) = d_1 + b_1 = d_1' + b_1'$ .

Without loss of generality, let  $d_1' \geq d_2'$  hold. Then  $d'$  and  $b'$  can be written as  $(d_1', d_1' - a_2)$  and  $(b_1', b_1' + a_2)$  for a certain  $a_2 > 0$ ,  $a_2 = d_1' - d_2'$ . We consider now the symmetric feasible set

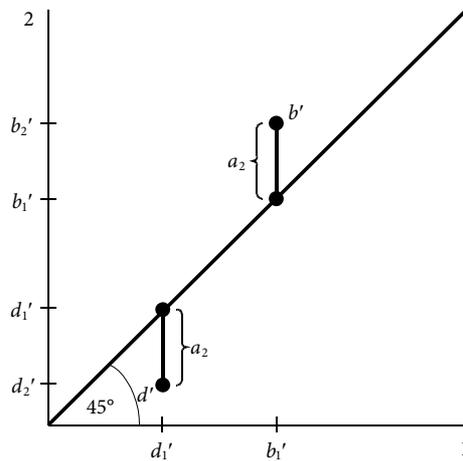
$$T = S' \cap \pi S' \subset S'. (T, (d_1', d_1'), (b_1', b_1')) \in C^2.$$

This means that  $d_1'$  and  $b_1'$  are arranged along the 45°-line like presented in Figure 3. Because of (iv),  $(d_1', b_1')$  is in  $S'$  and because of the comprehensiveness of  $S'$ ,  $(b_1', b_1')$  is still individually feasible, i.e.  $(S', (d_1', d_1'), (b_1', b_1'))$  is in  $B^2$ . The axiom of independence of compensation then induces

$$F(S', d', b') = F(S', (d_1', d_1'), (b_1', b_1')).$$

Restricted monotonicity implies

Figure 3



$$F(T, (d_1', d_1'), (b_1', b_1')) \leq F(S', (d_1', d_1'), (b_1', b_1'))$$

and because of the axiom symmetry,  $F(T, (d_1', d_1'), (b_1', b_1'))$  is symmetric, i.e. it lies on the 45°-line. In addition, it is an element of the Pareto frontier of  $S'$ . Therefore,  $F(S', (d_1', d_1'), (b_1', b_1'))$  has to be equal to  $F(T, (d_1', d_1'), (b_1', b_1'))$ . This means that  $F(S', d', b')$  is symmetric. This implies  $F_1(S, d, b) = F_2(S, d, b) + c$ . This together with Pareto optimality of  $F$  yields the result. ■

#### 4. Conclusions

We have presented a new axiomatic approach to the Nash-bargaining problem. In this approach we use a weak form of invariance of the solution: adding a constant utility to the utility function of each player does not change the solution. This is the only equivalent transformation of the utility function we consider. A reason for this weakening is the following: If we assume that the players have logarithmic utility functions in payoffs, then adding a constant utility which corresponds to multiplying payoffs by a constant factor remains to be the only operation that leaves the economic agreement invariant. In the model, however, no special utility function is used which might be motivated by empirical studies. We just weaken the invariance axiom. We have shown that a deeper analysis of the connection between payoffs and utilities might

lead to bargaining solution concepts that are good predictors for experimental or empirical data.

The axioms we required for our solution are Symmetry, Pareto Optimality and Restricted Monotonicity which are standard axioms in bargaining theory. We weakened the axiom of invariance under positive affine transformations to invariance under shifting. As the decisive axiom we introduced the axiom of independence of compensation which is based on equity considerations given the status quo and the bliss point as reference points. These axioms uniquely determine a bargaining rule. The last two axioms are motivated and supported by behavioral studies on bounded rationality.

The result of our approach is not far away from classical approaches if one considers solutions given in the Talmud as classical (cf. Aumann and Maschler 1985). Further support of our solution is given by an experimental study (Vogt and Albers 2001) whereas many experimental studies in bilateral negotiations show that many theoretical solution concepts do not lead to good predictions.

The claim of this paper is, however, not to have found the best solution to this problem, but to show that a more detailed look at the connection between monetary real world payoffs and the resulting utility might change results and give new results which are also suitable.

## References

- Albers W. (1997): Foundations of a theory of prominence in the decimal system, part I-V, Bielefeld: *IMW Working Papers* No. 265, 266, 269, 270 and 271.
- Allais M. (1953): La psychologie de l'homme rationnel devant le risque: critique des postulats et axiomes de l'école Américaine, *Econometrica* 21, 503–46.
- Aumann, R., Maschler, M. (1985): Game theoretic analysis of a bankruptcy problem from the Talmud, *Journal of Economic Theory* 36, 195–213.
- Bernoulli D. (1738): *Specimen theoriae novae de mensura sortis*, translated by L. Sommer (1954): Exposition of a new theory on the measurement of risk, *Econometrica* 22, 23–36.
- Chun Y. (1988): The equal-loss principle in bargaining problems, *Economics Letters* 26, 103–106.
- Fechner G.T. (1968): In *Sachen der Psychophysik*, Amsterdam: E.J. Bonset.
- Hokari T., Thomson, W. (2000): Bankruptcy and weighted generalizations of the Talmud rule, Working paper Kyoto University and University of Rochester.
- Kahneman D., Tversky A. (1979): Prospect theory: an analysis of decision under risk, *Econometrica* 47, 263–291.
- Kalai E., Smorodinsky M. (1975): Other solutions to Nash's bargaining problem, *Econometrica* 43, 513–518.

- Kalai E. (1977): Proportional solutions to bargaining situations: interpersonal utility comparisons, *Econometrica* 45, 1623–1630.
- Klemisch-Ahlert, M. (1996): *Bargaining in economic and ethical environments. An experimental study and normative solution concepts*, Springer Verlag Berlin Heidelberg.
- Nash J.F. (1950): The bargaining problem, *Econometrica* 18, 155–162.
- Nash J.F. (1953): Two person cooperative games, *Econometrica* 21, 128–140.
- Roth, A. E. (1979): *Axiomatic Models of Bargaining*, Springer Verlag, Berlin, Heidelberg.
- Sauermann H. (ed.) (1978): *Bargaining Behavior, Contributions to Experimental Economics*, Vol 7, Tübingen.
- Tietz, R. and Weber, H.-J. (1978): Decision Behavior in Multivariable Negotiations. In: Sauermann H. (ed.) *Bargaining Behavior, Contributions to Experimental Economics*, Vol 7, Tübingen.
- Thomson W. (1994): Cooperative Models of Bargaining, in: *Handbook of Game Theory*, R. J. Aumann, S. Hart (Hrsg.), Vol. 2, 1237–1284.
- Tversky A., Kahneman D. (1992): Advances in prospect theory: cumulative representation of uncertainty, *Journal of Risk and Uncertainty*, 297–232.
- Vogt B., Albers W. (1997): Equilibrium selection in 2x2 bimatrix games with preplay communication, Bielefeld, *IMW working paper* No 267.
- Vogt B., Albers W. (2001): Selection between Pareto-optimal outcomes in two-person bargaining with and without the right to make a proposal, *Homo Oeconomicus*, 77–90.
- Vogt B. (1999): Full information, hidden action and hidden information in principal-agent games, Bielefeld, *IMW working paper* No 315.